

FACTORIZATION IN MIXED NORM HARDY AND BMO SPACES

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ABSTRACT. Let $1 \leq p, q < \infty$ and $1 \leq r \leq \infty$. We show that the direct sum of mixed norm Hardy spaces $\left(\sum_n H_n^p(H_n^q)\right)_r$ and the sum of their dual spaces $\left(\sum_n H_n^p(H_n^q)^*\right)_r$ are both primary. We do so by using Bourgain's localization method and solving the finite dimensional factorization problem. In particular, we obtain that the spaces $\left(\sum_{n \in \mathbb{N}} H_n^1(H_n^s)\right)_r$, $\left(\sum_{n \in \mathbb{N}} H_n^s(H_n^1)\right)_r$, as well as $\left(\sum_{n \in \mathbb{N}} \text{BMO}_n(H_n^s)\right)_r$ and $\left(\sum_{n \in \mathbb{N}} H_n^s(\text{BMO}_n)\right)_r$, $1 < s < \infty$, $1 \leq r \leq \infty$, are all primary.

1. INTRODUCTION

Let \mathcal{D} denote the collection of dyadic intervals on the unit interval, which is given by

$$\mathcal{D} = \{[k2^{-n}, (k+1)2^{-n}) : n, k \in \mathbb{N}_0, 0 \leq k \leq 2^n - 1\}.$$

The dyadic intervals are nested, i.e. if $I, J \in \mathcal{D}$, then $I \cap J \in \{I, J, \emptyset\}$. For $I \in \mathcal{D}$ we let $|I|$ denote the length of the dyadic interval I . The Carleson constant $\llbracket \mathcal{C} \rrbracket$ of a collection $\mathcal{C} \subset \mathcal{D}$ is given by

$$\llbracket \mathcal{C} \rrbracket = \sup_{I \in \mathcal{C}} \frac{1}{|I|} \sum_{\substack{J \in \mathcal{C} \\ J \subset I}} |J|.$$

Let $I \in \mathcal{D}$ and $I \neq [0, 1)$, then \tilde{I} is the unique dyadic interval satisfying $\tilde{I} \supset I$ and $|\tilde{I}| = 2|I|$. Given $N_0 \in \mathbb{N}_0$ we define

$$\mathcal{D}_{N_0} = \{I \in \mathcal{D} : |I| = 2^{-N_0}\} \quad \text{and} \quad \mathcal{D}^{N_0} = \{I \in \mathcal{D} : |I| \geq 2^{-N_0}\}.$$

Let h_I be the L^∞ -normalized Haar function supported on $I \in \mathcal{D}$; that is, h_I is +1 on the left half of I , it is -1 on the right half of I , and zero otherwise. For $1 \leq p < \infty$, the *Hardy space* H^p is the completion of

$$\text{span}\{h_I : I \in \mathcal{D}\}$$

under the square function norm

$$\|f\|_{H^p} = \left(\int_0^1 \left(\sum_I |a_I|^2 h_I^2(x) \right)^{p/2} dx \right)^{1/p}, \quad (1.1)$$

where $f = \sum_I a_I h_I$.

Let $\mathcal{R} = \{I \times J : I, J \in \mathcal{D}\}$ denote the collection of dyadic rectangles contained in the unit square, and define the bi-parameter L^∞ -normalized Haar system by

$$h_{I \times J}(x, y) = h_I(x)h_J(y), \quad I \times J \in \mathcal{R}, \quad x, y \in [0, 1).$$

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For $1 \leq p, q < \infty$, the *mixed-norm Hardy space* $H^p(H^q)$ is the completion of

$$\text{span}\{h_{I \times J} : I \times J \in \mathcal{R}\}$$

under the square function norm

$$\|f\|_{H^p(H^q)} = \left(\int_0^1 \left(\int_0^1 \left(\sum_{I \times J} |a_{I \times J}|^2 h_{I \times J}^2(x, y) \right)^{q/2} dy \right)^{p/q} dx \right)^{1/p}, \quad (1.2)$$

where $f = \sum_{I \times J} a_{I \times J} h_{I \times J}$. Given $m, n \in \mathbb{N}$, we define the space $H_m^p(H_n^q)$ by

$$H_m^p(H_n^q) = \text{span}\{h_{I \times J} : I \in \mathcal{D}^m, J \in \mathcal{D}^n\},$$

equipped with the norm $\|\cdot\|_{H^p(H^q)}$.

For the following elementary and well known facts for which we refer to [8, 9, 4, 13, 10, 11] as sources:

- ▷ $(h_{I \times J})_{I \times J \in \mathcal{R}}$ is an unconditional basis of $H^p(H^q)$, called the *bi-parameter Haar system*. This basis is L^∞ -normalized and not normalized in $H^p(H^q)$; in fact, we have $\|h_{I \times J}\|_{H^p(H^q)} = |I|^{1/p} |J|^{1/q}$.
- ▷ Let $1 \leq p, q < \infty$ and let $H^p(H^q)^*$ denote the dual space of $H^p(H^q)$ with the usual operator norm given by

$$\|g\|_{H^p(H^q)^*} = \sup\{|\langle g, f \rangle| : \|f\|_{H^p(H^q)} \leq 1\}. \quad (1.3)$$

- ▷ Since $h_{I \times J}$, $I \times J$ is a Schauder basis in $H^p(H^q)$, we canonically identify the elements $g \in H^p(H^q)^*$ with the sequence $(\langle g, h_{I \times J} \rangle)_{I \times J}$. Moreover, as $h_{I \times J}$, $I \times J$ is a 1-unconditional basis in $H^p(H^q)$, the norm of $(|\langle g, h_{I \times J} \rangle|)_{I \times J}$ is equal to the norm of $(\langle g, h_{I \times J} \rangle)_{I \times J}$, see [8, Chapter 1].
- ▷ We naturally identify $h_{I_0 \times J_0}$ as an element of $H^p(H^q)^*$ by the following definition: $\langle h_{I_0 \times J_0}, h_{I \times J} \rangle = |I \times J|$ if $I \times J = I_0 \times J_0$, and $\langle h_{I_0 \times J_0}, h_{I \times J} \rangle = 0$ if $I \times J \neq I_0 \times J_0$.
- ▷ Let $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$. Since for any finite linear combination of Haar functions f we have

$$C_{p,q}^{-1} \|f\|_{L^p(L^q)} \leq \|f\|_{H^p(H^q)} \leq C_{p,q} \|f\|_{L^p(L^q)},$$

the identity operator provides an isomorphism between $H^p(H^q)$ and $L^p(L^q)$. Hence, the dual of $H^p(H^q)$ identifies with $H^{p'}(H^{q'})$. Similarly, for the limiting cases we have $H^1(H^q)^* = \text{BMO}(H^{q'})$, $H^p(H^1)^* = H^{q'}(\text{BMO})$ and $H^1(H^1)^* = \text{BMO}(\text{BMO})$. See [9] and also [10].

- ▷ The isomorphisms that identify the duals of the mixed norm Hardy spaces $H^p(H^q)$, $1 \leq p, q < \infty$ with the spaces mentioned above may or may not depend on p and q (the uncertainty stems from not specifying the norm of BMO). Since the constants in our results do not depend on p or q , but some of the proofs involve the dual space of $H^p(H^q)$, we have to be careful not to introduce dependencies on p and q this way. As we will see, all the proofs are carried out by strictly using the dual norm of $H^p(H^q)$ specified in (1.3), thereby avoiding the problem of introducing p or q dependencies in the estimates.

Let $k, m, n \in \mathbb{N}$ and $1 \leq p, q < \infty$. Let $(b_i : 1 \leq i \leq k)$ denote a block basis of bi-parameter Haar functions in $H_m^p(H_n^q)$, and let $(b_i^* : 1 \leq i \leq k)$ denote the bi-orthogonal functions, i.e. $\langle b_i^*, b_i \rangle = 1$, and $\langle b_i^*, b_j \rangle = 0$, if $i \neq j$. We say an operator $T : H_m^p(H_n^q) \rightarrow H_m^p(H_n^q)$ has *large diagonal* with respect to the system $(b_i : 1 \leq i \leq k)$, if there exists a $\delta > 0$ such that $|\langle b_i^*, T b_i \rangle| > \delta$, for all $1 \leq i \leq k$, and δ does not depend on any of m, n, k, p, q .

We will briefly state the version of Pelczyński's decomposition method that we will use here: Let X and Y be Banach spaces so that X is isomorphic to a complemented subspace of Y , and vice versa. If X is such that X is isomorphic to $(\sum X)_r$ for some $1 \leq r \leq \infty$, then X is isomorphic to Y .

The main object that we will study are the spaces $(\sum_{m,n \in \mathbb{N}} H_m^p(H_n^q))_r$, $1 \leq p, q < \infty$ and $1 \leq r \leq \infty$. They are defined as follows:

$$(\sum_{m,n \in \mathbb{N}} H_m^p(H_n^q))_r = \{f = (f_{m,n})_{m,n \in \mathbb{N}} : f_{m,n} \in H_m^p(H_n^q), \|f\|_r < \infty\}, \quad (1.4)$$

equipped with the norm $\|f\|_r$ given by

$$\|f\|_r = \left(\sum_{m,n \in \mathbb{N}} \|f_{m,n}\|_{H^p(H^q)}^r \right)^{1/r}, \text{ if } r < \infty, \text{ and } \|f\|_\infty = \sup_{m,n} \|f_{m,n}\|_{H^p(H^q)}. \quad (1.5)$$

Naturally, the question arises how many non-isomorphic spaces are defined by (1.4) and (1.5). In Proposition 5.5 we assert that

$$(\sum_{m,n \in \mathbb{N}} H_m^p(H_n^q))_r \text{ is isometrically isomorphic to } (\sum_{n \in \mathbb{N}} H_n^p(H_n^q))_r, \quad (1.6)$$

and that by a variant of Pitt's theorem (see Theorem 5.6) the spaces

$$(\sum_{n \in \mathbb{N}} H_n^p(H_n^q))_r \text{ and } (\sum_{n \in \mathbb{N}} H_n^p(H_n^q))_s \text{ are not isomorphic for } 1 \leq r \neq s \leq \infty. \quad (1.7)$$

2. MAIN RESULTS

Here, we state the main results Theorem 2.1 and Theorem 2.2 and describe the concept of proof. Their respective proofs are carried out in Section 5.

Theorem 2.1. *Let $1 \leq p, q < \infty$ and $1 \leq r \leq \infty$, and for all $n \in \mathbb{N}$ let X_n denote the space $H_n^p(H_n^q)$ or its dual $H_n^p(H_n^q)^*$. For any $\eta > 0$ and any operator $T : (\sum_{n \in \mathbb{N}} X_n)_r \rightarrow (\sum_{n \in \mathbb{N}} X_n)_r$, there exist operators $R, S : (\sum_{n \in \mathbb{N}} X_n)_r \rightarrow (\sum_{n \in \mathbb{N}} X_n)_r$ such that*

$$\begin{array}{ccc} (\sum_{n \in \mathbb{N}} X_n)_r & \xrightarrow{\text{Id}} & (\sum_{n \in \mathbb{N}} X_n)_r \\ S \downarrow & & \uparrow R \\ (\sum_{n \in \mathbb{N}} X_n)_r & \xrightarrow{H} & (\sum_{n \in \mathbb{N}} X_n)_r \end{array} \quad (2.1)$$

for $H = T$ or $H = \text{Id} - T$ and $\|R\|\|S\| \leq 2 + \eta$.

There are several factorization results of the form (2.1) regarding bi-parameter Hardy spaces or their duals; among them are the following:

- ▷ $H^p(H^q)$ for $1 < p, q < \infty$, see [4].
- ▷ $H^1(H^1)$, see [10].
- ▷ $(\sum_{n \in \mathbb{N}} H_n^1(H_n^1))_r$ and $(\sum_{n \in \mathbb{N}} \text{BMO}_n(\text{BMO}_n))_r$, $1 \leq r \leq \infty$, see [7].

We remark that the latter result is only stated explicitly for the space $(\sum_{n \in \mathbb{N}} \text{BMO}_n(\text{BMO}_n))_\infty$. The other assertions follow immediately from reasonable modifications of the proof given in [7].

Theorem 2.1 is an extension of the above list. Specifically, Theorem 2.1 yields factorization results (with possibly different constants as discussed in the introduction) for the following spaces:

$$(\sum_{n \in \mathbb{N}} H_n^1(H_n^s))_r, \quad (\sum_{n \in \mathbb{N}} H_n^s(H_n^1))_r, \quad (\sum_{n \in \mathbb{N}} \text{BMO}_n(H_n^s))_r, \quad (\sum_{n \in \mathbb{N}} H_n^s(\text{BMO}_n))_r,$$

where $1 < s < \infty$, $1 \leq r \leq \infty$. The proof of Theorem 2.1 is based on Bourgain's localization method [3] and consists of the following three major steps:

- ▷ Reduction to diagonal operators.
- ▷ Proving the following quantitative factorization problem: For all $n \in \mathbb{N}$ and $\Gamma, \eta > 0$ there exists an integer $N = N(n, \Gamma, \eta)$ so that the following holds: For any operator $D : X_N \rightarrow X_N$ with $\|D\| \leq \Gamma$ there exist operators R, S so that for either $H = D$ or $H = \text{Id}_{X_n} - D$ we have that

$$\begin{array}{ccc} X_n & \xrightarrow{\text{Id}} & X_n \\ S \downarrow & & \uparrow R \\ X_N & \xrightarrow{H} & X_N \end{array} \quad \text{where } \|R\| \|S\| \leq 2 + \eta. \quad (2.2)$$

- ▷ Glueing the finite dimensional pieces (2.2) together to obtain the factorization diagram (2.1).

Before we come to the next result, let us recall the notion of a primary Banach space, see e.g. [8]: A Banach space X is *primary* if for every bounded projection $Q : X \rightarrow X$, either $Q(X)$ or $(\text{Id} - Q)(X)$ is isomorphic to X . An immediate consequence of Theorem 2.1 is the subsequent Theorem 2.2.

Theorem 2.2. *Let $1 \leq p, q < \infty$ and $1 \leq r \leq \infty$. Then $(\sum_{n \in \mathbb{N}} H_n^p(H_n^q))_r$ and $(\sum_{n \in \mathbb{N}} H_n^p(H_n^q)^*)_r$ are primary. In particular, the following spaces are primary:*

$$\left(\sum_{n \in \mathbb{N}} H_n^1(H_n^s) \right)_r, \quad \left(\sum_{n \in \mathbb{N}} H_n^s(H_n^1) \right)_r, \quad \left(\sum_{n \in \mathbb{N}} \text{BMO}_n(H_n^s) \right)_r, \quad \left(\sum_{n \in \mathbb{N}} H_n^s(\text{BMO}_n) \right)_r,$$

where $1 < s < \infty$.

Note that if $p = q = 1$, Theorem 2.2 follows from [7].

3. THE LOCAL PRODUCT CONDITIONS

In Section 3.1 and Section 3.2 we discuss conditions (the local product conditions (P1)–(P4)) under which a block basis of the bi-parameter Haar system is equivalent to the bi-parameter Haar system, and that the orthogonal projection onto this block basis is bounded in $H^p(H^q)$. Section 3.1 and 3.2 are a compilation of definitions and results of [6]. Section 3.3 is new; it contains the result that reiterating the local product conditions yields again the local product conditions. The local product conditions were modeled after Capon's conditions isolated in [4].

3.1. Statement of the local product conditions.

Let $\mathcal{A} \subset \mathcal{R}$ be an index set. For each $R \in \mathcal{A}$ let $\mathcal{X}_R, \mathcal{Y}_R \subset \mathcal{D}$ denote non-empty collections of dyadic intervals that define the collection of dyadic rectangles \mathcal{B}_R by

$$\mathcal{B}_R = \mathcal{X}_R \times \mathcal{Y}_R = \{K \times L : K \in \mathcal{X}_R, L \in \mathcal{Y}_R\}, \quad R \in \mathcal{A}, \quad (3.1)$$

see Figure 1 and 2. For all $R \in \mathcal{A}$ and $x, y \in [0, 1)$ we define

$$b_R(x, y) = \sum_{K \times L \in \mathcal{B}_R} h_{K \times L}(x, y) = \left(\sum_{K \in \mathcal{X}_R} h_K(x) \right) \left(\sum_{L \in \mathcal{Y}_R} h_L(y) \right). \quad (3.2)$$

For the second equality in (3.2) see (3.1). We call $(b_R : R \in \mathcal{R})$ the *block basis generated by* $(\mathcal{B}_R : R \in \mathcal{R})$.

We now introduce some notation. For $R \in \mathcal{A}$ we set

$$X_R = \bigcup \{K : K \in \mathcal{X}_R\} \quad \text{and} \quad Y_R = \bigcup \{L : L \in \mathcal{Y}_R\}. \quad (3.3)$$

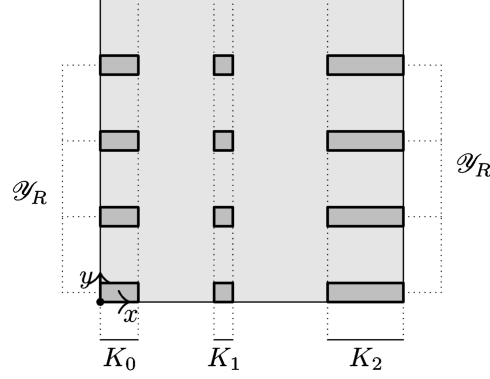


FIGURE 1. Given a dyadic index rectangle $R \in \mathcal{A}$, this figure depicts the collection of dark gray rectangles \mathcal{B}_R , which are contained in the light gray unit square. \mathcal{B}_R is of the form $\mathcal{B}_R = \mathcal{X}_R \times \mathcal{Y}_R$, where $\mathcal{X}_R = \{K_0, K_1, K_2\}$.

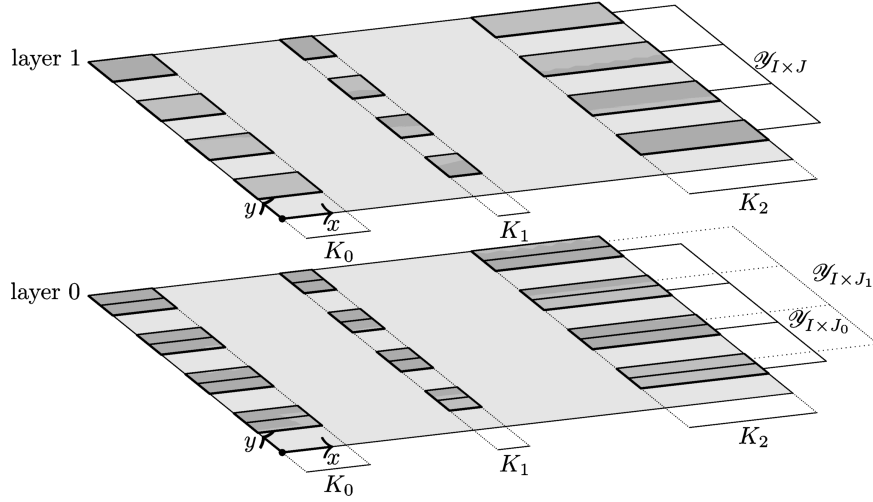


FIGURE 2. The dyadic index rectangles $I \times J$, $I \times J_0$ and $I \times J_1$ in \mathcal{A} are such that $J_0 \cup J_1 = J$ and $J_0 \cap J_1 = \emptyset$. This figure depicts the collections $\mathcal{B}_{I \times J} = \mathcal{X}_{I \times J} \times \mathcal{Y}_{I \times J}$ in the top layer (see also Figure 1), and $\mathcal{B}_{I \times J_0} = \mathcal{X}_{I \times J_0} \times \mathcal{Y}_{I \times J_0}$ and $\mathcal{B}_{I \times J_1} = \mathcal{X}_{I \times J_1} \times \mathcal{Y}_{I \times J_1}$ in the bottom layer, where $\mathcal{X}_{I \times J} = \{K_0, K_1, K_2\}$. Each interval in $\mathcal{Y}_{I \times J}$ is split in two intervals, which are then placed into $\mathcal{Y}_{I \times J_0}$ and $\mathcal{Y}_{I \times J_1}$, respectively.

For each $I_0 \times J_0 \in \mathcal{A}$ we take the following unions:

$$X_{I_0} = \bigcup \{X_{I_0 \times J} : I_0 \times J \in \mathcal{A}\}, \quad Y_{J_0} = \bigcup \{Y_{I \times J_0} : I \times J_0 \in \mathcal{A}\}. \quad (3.4)$$

By (3.4) we have that for all $I \times J \in \mathcal{A}$

$$X_{I \times J} \subset X_I \quad \text{and} \quad Y_{I \times J} \subset Y_J. \quad (3.5)$$

We say that $\{\mathcal{B}_{I \times J} : I \times J \in \mathcal{A}\}$ satisfies the *local product conditions* with constants $C_X, C_Y > 0$, if the following four conditions (P1), (P2), (P3) and (P4) hold.

- (P1) For all $R \in \mathcal{A}$ the collection \mathcal{B}_R consists of pairwise disjoint dyadic rectangles, and for all $R_0, R_1 \in \mathcal{A}$ with $R_0 \neq R_1$ we have $\mathcal{B}_{R_0} \cap \mathcal{B}_{R_1} = \emptyset$.
- (P2) For all $I \times J, I_0 \times J_0, I_1 \times J_1 \in \mathcal{A}$ with $I_0 \cap I_1 = \emptyset$, $I_0 \cup I_1 \subset I$ and $J_0 \cap J_1 = \emptyset$, $J_0 \cup J_1 \subset J$ we have the inclusions

$$\begin{aligned} X_{I_0} \cap X_{I_1} &= \emptyset & X_{I_0} \cup X_{I_1} &\subset X_I, \\ Y_{J_0} \cap Y_{J_1} &= \emptyset & Y_{J_0} \cup Y_{J_1} &\subset Y_J. \end{aligned}$$

- (P3) For all $I \times J \in \mathcal{A}$, we have

$$C_X^{-1}|I| \leq |X_I| \leq C_X|I| \quad \text{and} \quad C_Y^{-1}|J| \leq |Y_J| \leq C_Y|J|.$$

- (P4) For all $I_0 \times J_0, I \times J \in \mathcal{A}_1$ with $I_0 \times J_0 \subset I \times J$ and for every $K \in \mathcal{X}_{I \times J}$ and $L \in \mathcal{Y}_{I_0 \times J_0}$, we have

$$\frac{|K \cap X_{I_0}|}{|K|} \geq C_X^{-1} \frac{|X_{I_0}|}{|X_I|} \quad \text{and} \quad \frac{|L \cap Y_{J_0}|}{|L|} \geq C_Y^{-1} \frac{|Y_{J_0}|}{|Y_J|}.$$

3.2. Implications from the local product conditions.

The local product conditions (P1)–(P4) ensure that the block basis $(b_R : R \in \mathcal{A})$ given by (3.2) is equivalent to the Haar system $(h_R : R \in \mathcal{A})$ in $H^p(H^q)$, for $1 < p, q < \infty$, and that the orthogonal projection $Q^{(\varepsilon)}$ (see (3.9) below) onto $(b_R : R \in \mathcal{A})$ is bounded in $H^p(H^q)$, $1 < p, q < \infty$.

In particular, Theorem 3.1 includes the following endpoints:

$$H^1(H^s), \quad H^s(H^1), \quad H^s(\text{BMO}), \quad \text{BMO}(H^s), \quad 1 < s < \infty.$$

Theorem 3.1 is taken from [6].

Theorem 3.1. *Let $1 \leq p, q < \infty$. Assume that $(\mathcal{B}_R : R \in \mathcal{R})$ satisfies the local product conditions (P1)–(P4) with constants C_X and C_Y . Let $\varepsilon = (\varepsilon_Q : Q \in \mathcal{R})$ be a scalar sequence with $|\varepsilon_Q| = 1$, and let the block basis of the bi-parameter Haar system $b_R^{(\varepsilon)}$ be given by*

$$b_R^{(\varepsilon)} = \sum_{Q \in \mathcal{B}_R} \varepsilon_Q h_Q, \quad R \in \mathcal{R}. \quad (3.6)$$

Then the following assertions are true:

- (i) *For all sequences of scalars a_R , $R \in \mathcal{R}$, we have that*

$$C^{-1} \left\| \sum_R a_R h_R \right\| \leq \left\| \sum_R a_R b_R^{(\varepsilon)} \right\| \leq C \left\| \sum_R a_R h_R \right\|. \quad (3.7)$$

The above norms are either all the norm $\|\cdot\|_{H^p(H^q)}$, or they are all the norm $\|\cdot\|_{H^p(H^q)^}$.*

- (ii) *The orthogonal projection $Q^{(\varepsilon)}$ given by*

$$Q^{(\varepsilon)} f = \sum_{R \in \mathcal{R}} \frac{\langle b_R^{(\varepsilon)}, f \rangle}{\|b_R^{(\varepsilon)}\|_2^2} b_R^{(\varepsilon)} \quad (3.8)$$

satisfies the estimates

$$\begin{aligned} \|Qf\|_{H^p(H^q)} &\leq C \|f\|_{H^p(H^q)}, \quad f \in H^p(H^q), \\ \|Qf\|_{H^p(H^q)^*} &\leq C \|f\|_{H^p(H^q)^*}, \quad f \in H^p(H^q)^*. \end{aligned} \quad (3.9)$$

There exists a universal integer k such that $C \leq C_X^k C_Y^k$.

We remark that by (3.1), we can rewrite (3.6) in the following ways:

$$\begin{aligned} b_R^{(\varepsilon)}(x, y) &= \sum_{K \in \mathcal{X}_R} h_K(x) \sum_{L \in \mathcal{Y}_R} \varepsilon_{K \times L} h_L(y), \quad R \in \mathcal{R}, \\ b_R^{(\varepsilon)}(x, y) &= \sum_{L \in \mathcal{Y}_R} h_L(y) \sum_{K \in \mathcal{X}_R} \varepsilon_{K \times L} h_K(x), \quad R \in \mathcal{R}. \end{aligned} \quad (3.10)$$

3.3. Reiterating the local product conditions.

Let (e_I) and (f_J) denote block bases of the one parameter Haar system, such that $(e_I \otimes f_J)$ satisfies (P1)–(P4). Moreover, we will assume that the regularity assumptions Lemma 3.2 (i) and (ii) are satisfied. Assumption (i) really is a one-parameter version of (P1). The more interesting assumption is (ii), which says that the inclusion of two index intervals $I_0 \subset I$ (respectively $J_0 \subset J$) implies the inclusion of each $E_0 \in \mathcal{E}_{I_0}$ in some $E \in \mathcal{E}_I$ (respectively of each $F_0 \in \mathcal{F}_{J_0}$ in some $F \in \mathcal{F}_J$). Lemma 3.2 tells us that if one uses $(e_I \otimes f_J)$ instead of the bi-parameter Haar system $(h_I \otimes h_J)$ to build a bi-parameter block basis according to (P1)–(P4), then the result is a block basis of the bi-parameter Haar system satisfying the local product conditions (P1)–(P4).

Lemma 3.2. *Let \mathcal{A} be a collection of index rectangles. Let*

$$(\mathcal{E}_I \times \mathcal{F}_J : I \times J \in \mathcal{A})$$

be a sequence of dyadic rectangles satisfying (P1)–(P4) with constants C_E and C_F . Moreover, we assume the following:

- (i) *For all $I \times J \in \mathcal{A}$ the collections \mathcal{E}_I and \mathcal{F}_J each consist of pairwise disjoint intervals, and $\mathcal{E}_{I_0} \cap \mathcal{E}_{I_1} = \emptyset$ for all $I_0 \times J_0, I_1 \times J_1 \in \mathcal{A}$ with $I_0 \neq I_1$, and $\mathcal{F}_{J_0} \cap \mathcal{F}_{J_1} = \emptyset$, for all $I_0 \times J_0, I_1 \times J_1 \in \mathcal{A}$ with $J_0 \neq J_1$.*
- (ii) *Whenever $I_0 \times J_0, I \times J \in \mathcal{A}$ with $I_0 \subset I$ and $J_0 \subset J$, then*

$$\begin{aligned} &\text{for all } E_0 \in \mathcal{E}_{I_0} \text{ there exists an } E \in \mathcal{E}_I \text{ such that } E_0 \subset E, \\ &\text{for all } F_0 \in \mathcal{F}_{J_0} \text{ there exists an } F \in \mathcal{F}_J \text{ such that } F_0 \subset F. \end{aligned}$$

Let

$$\left(\mathcal{B}_{E \times F} : E \times F \in \bigcup_{I \times J \in \mathcal{A}} \mathcal{E}_I \times \mathcal{F}_J \right)$$

be a sequence of collections of dyadic rectangles satisfying the local product conditions (P1)–(P4) with constants C_X and C_Y . For each $I \times J \in \mathcal{A}$ define the collection $\tilde{\mathcal{B}}_{I \times J}$ by

$$\tilde{\mathcal{B}}_{I \times J} = \bigcup_{\substack{E \in \mathcal{E}_I \\ F \in \mathcal{F}_J}} \mathcal{B}_{E \times F}.$$

Then the sequence of collections $(\tilde{\mathcal{B}}_{I \times J})$ satisfies the local product conditions (P1)–(P4) with constants $C_E C_X^3$ and $C_F C_Y^3$.

Remark 3.3. Consequently, Theorem 3.1 applies to the collections $\tilde{\mathcal{B}}_R$ and the block basis of the bi-parameter Haar system $\tilde{b}_R^{(\varepsilon)}$ given by

$$\tilde{b}_{I \times J}^{(\varepsilon)} = \sum_{Q \in \tilde{\mathcal{B}}_{I \times J}} \varepsilon_Q h_Q = \sum_{E \in \mathcal{E}_I} \sum_{F \in \mathcal{F}_J} b_{E \times F}^{(\varepsilon)}, \quad I \times J \in \mathcal{R},$$

where the block basis $(b_{E \times F}^{(\varepsilon)})$ is given by

$$b_{E \times F}^{(\varepsilon)} = \sum_{Q \in \mathcal{B}_{E \times F}} \varepsilon_Q h_Q, \quad E \times F \in \bigcup_{I \times J \in \mathcal{R}} \mathcal{E}_I \times \mathcal{F}_J.$$

Proof of Lemma 3.2. Within this proof, we shall make use of the following convention. Whenever there is an identifier of an object which uses the script font (i.e. \mathcal{X}), then the same identifier in roman font denotes its pointset (i.e. $Z = \bigcup \mathcal{Z}$). As in Section 3.1, we write

$$\mathcal{B}_{E \times F} = \mathcal{X}_{E \times F} \times \mathcal{Y}_{E \times F}, \quad E \times F \in \bigcup_{I \times J \in \mathcal{A}} \mathcal{E}_I \times \mathcal{F}_J.$$

Firstly, we define the collections of dyadic intervals $\widetilde{\mathcal{X}}_{I \times J}$ by

$$\widetilde{\mathcal{X}}_{I \times J} = \bigcup_{\substack{E \in \mathcal{E}_I \\ F \in \mathcal{F}_J}} \mathcal{X}_{E \times F}, \quad I \times J \in \mathcal{A}. \quad (3.11)$$

Secondly, we define the collections of dyadic intervals $\widetilde{\mathcal{Y}}_{I \times J}$ by

$$\widetilde{\mathcal{Y}}_{I \times J} = \bigcup_{\substack{E \in \mathcal{E}_I \\ F \in \mathcal{F}_J}} \mathcal{Y}_{E \times F}, \quad I \times J \in \mathcal{A}. \quad (3.12)$$

Thirdly, observe that the following identity is true:

$$\widetilde{\mathcal{B}}_{I \times J} = \widetilde{\mathcal{X}}_{I \times J} \times \widetilde{\mathcal{Y}}_{I \times J}, \quad I \times J \in \mathcal{A}. \quad (3.13)$$

We now show (P1) for $(\widetilde{\mathcal{B}}_{I \times J})$. Let $I_0 \times J_0, I_1 \times J_1 \in \mathcal{A}$ and assume that $\mathcal{B}_{I_0 \times J_0} \cap \mathcal{B}_{I_1 \times J_1} \neq \emptyset$. Then there exist dyadic intervals $E_0 \in \mathcal{E}_{I_0}$, $E_1 \in \mathcal{E}_{I_1}$ and $F_0 \in \mathcal{F}_{J_0}$, $F_1 \in \mathcal{F}_{J_1}$ such that $\mathcal{B}_{E_0 \times F_0} \cap \mathcal{B}_{E_1 \times F_1} \neq \emptyset$. Since $(\mathcal{B}_{E \times F})$ satisfies (P1), we infer $E_0 = E_1$ and $F_0 = F_1$. Thus, $\mathcal{E}_{I_0} \cap \mathcal{E}_{I_1} \neq \emptyset$ and $\mathcal{F}_{J_0} \cap \mathcal{F}_{J_1} \neq \emptyset$, which implies $I_0 = I_1$ and $J_0 = J_1$. Now, let $I \times J \in \mathcal{A}$ and assume that there are $K_0 \times L_0, K_1 \times L_1 \in \widetilde{\mathcal{B}}_{I \times J}$ such that $K_0 \times L_0 \cap K_1 \times L_1 \neq \emptyset$, i.e. $K_0 \cap K_1 \neq \emptyset$ and $L_0 \cap L_1 \neq \emptyset$. Clearly, there exist $E_0, E_1 \in \mathcal{E}_I$ and $F_0, F_1 \in \mathcal{F}_J$ so that $K_i \in \mathcal{X}_{E_i \times F_i}$ as well as $L_i \in \mathcal{Y}_{E_i \times F_i}$, $i = 0, 1$. This implies $X_{E_0} \cap X_{E_1} \supset X_{E_0 \times F_0} \cap X_{E_1 \times F_1} \neq \emptyset$ and $Y_{F_0} \cap Y_{F_1} \supset Y_{E_0 \times F_0} \cap Y_{E_1 \times F_1} \neq \emptyset$. Hence, by (P2) for the sequence of collections $(\mathcal{B}_{E \times F})$, we obtain $E_0 \cap E_1 \neq \emptyset$ and $F_0 \cap F_1 \neq \emptyset$. Since each of the two collections \mathcal{E}_I and \mathcal{F}_J consists of pairwise disjoint dyadic intervals, we note $E_0 = E_1$ and $F_0 = F_1$. Thus, $K_0 \times L_0 \cap K_1 \times L_1 \neq \emptyset$ and $K_i \times L_i \in \mathcal{B}_{E_0 \times F_0}$, $i = 0, 1$, so by (P1) we have that $K_0 \times L_0 = K_1 \times L_1$.

Next, we prove that $(\widetilde{\mathcal{B}}_{I \times J})$ has property (P2). To this end, let $I_k \times J_k \in \mathcal{A}$, $k = 0, 1$. We now show that

$$\widetilde{X}_{I_0} \cap \widetilde{X}_{I_1} \neq \emptyset \quad \text{implies} \quad I_0 \cap I_1 \neq \emptyset. \quad (3.14)$$

Let $\widetilde{X}_{I_0} \cap \widetilde{X}_{I_1} \neq \emptyset$. By (3.11) and (3.4) we obtain

$$\widetilde{X}_{I_k} = \bigcup_{E \in \mathcal{E}_{I_k}} X_E, \quad k = 0, 1, \quad (3.15)$$

thus we can find dyadic intervals $E_k \in \mathcal{E}_{I_k}$, $k = 0, 1$, such that $X_{E_0} \cap X_{E_1} \neq \emptyset$. But then, by (P2) for X_{E_0}, X_{E_1} , we have that $E_0 \cap E_1 \neq \emptyset$, and therefore $\mathcal{E}_{I_0} \cap \mathcal{E}_{I_1} \neq \emptyset$. By (P2) for $\mathcal{E}_{I_0}, \mathcal{E}_{I_1}$ we obtain $I_0 \cap I_1 \neq \emptyset$, which proves (3.14). Next, we prove

$$X_{I_0} \subset X_{I_1} \quad \text{whenever} \quad I_0 \subset I_1. \quad (3.16)$$

Let $I_0 \subset I_1$. By (P2) for E_{I_k} , $k = 0, 1$, we have that $E_{I_0} \subset E_{I_1}$. By (ii), we obtain that for all $E_0 \in \mathcal{E}_{I_0}$ there is an $E_1 \in \mathcal{E}_{I_1}$ such that $E_0 \subset E_1$. (P2) for X_{E_k} , $k = 0, 1$, implies $X_{E_0} \subset X_{E_1}$. Thus, we obtain from (3.15) that (3.16) holds. The respective proof for Y_{J_0}, Y_{J_1} is repeating the above proof for X_{I_0}, X_{I_1} , with Y replacing X , F replacing E and I replacing J .

Next, we will prove (P3). Again, since the proof for X_I and Y_J is completely analogous, we will prove (P3) only for X_I . Let $I \times J \in \mathcal{A}$. By (3.15), we have that

$$\tilde{X}_I = \bigcup_{E \in \mathcal{E}_I} X_E.$$

By (i) for \mathcal{E}_I and (P3) for X_E , $E \in \mathcal{E}_I$ and the above identity, we obtain

$$C_X^{-1}|E_I| \leq C_X^{-1} \sum_{E \in \mathcal{E}_I} |E| \leq |\tilde{X}_I| = \sum_{E \in \mathcal{E}_I} |X_E| \leq C_X \sum_{E \in \mathcal{E}_I} |E| = C_X|E_I|.$$

By (P3) we have that $C_X^{-1}|I| \leq |E_I| \leq C_X|I|$, which combined with the above estimate shows

$$C_E^{-1}C_X^{-1}|I| \leq |\tilde{X}_I| \leq C_EC_X|I|.$$

We note that (P3) holds with constants C_EC_X and $C_FC_Y^2$, respectively.

Finally, we will show that (P4) holds for $\mathcal{B}_{I \times J}$. For brevity, we will only show the estimates concerning X_{I_0} . The estimates for Y_{J_0} follow by replacing the proper characters in the proof given below. Let $I_0 \times J_0, I \times J \in \mathcal{A}$ with $I_0 \times J_0 \subset I \times J$, and let $K \in \mathcal{X}_{I \times J}$. By (3.11), there exist $E \in \mathcal{E}_I$ and $F \in \mathcal{F}_J$ such that $K \in \mathcal{X}_{E \times F}$. By (3.15), (i) and (P2) for X_{E_0} , we obtain that

$$\frac{|K \cap \tilde{X}_{I_0}|}{|K|} = \sum_{\substack{E_0 \in \mathcal{E}_{I_0} \\ E_0 \subset E}} \frac{|K \cap X_{E_0}|}{|K|} \geq \sum_{\substack{E_0 \in \mathcal{E}_{I_0} \\ E_0 \subset E}} \frac{|K \cap X_{E_0}|}{|K|}.$$

Using (P4) for $E_0 \subset E$, $K \in \mathcal{X}_{E \times F}$, we obtain

$$\frac{|K \cap \tilde{X}_{I_0}|}{|K|} \geq C_X^{-1} \sum_{\substack{E_0 \in \mathcal{E}_{I_0} \\ E_0 \subset E}} \frac{|X_{E_0}|}{|X_E|} \geq C_X^{-3} \sum_{\substack{E_0 \in \mathcal{E}_{I_0} \\ E_0 \subset E}} \frac{|E_0|}{|E|},$$

where the latter estimate follows from (P3) for X_{E_0} and X_E . Using the hypothesis (i) yields

$$\frac{|K \cap \tilde{X}_{I_0}|}{|K|} \geq C_X^{-3} \frac{|E \cap E_{I_0}|}{|E|}.$$

Invoking (P4) for $\frac{|E \cap E_{I_0}|}{|E|}$ gives

$$\frac{|K \cap \tilde{X}_{I_0}|}{|K|} \geq C_E^{-1}C_X^{-3} \frac{|I_0|}{|I|}.$$

We note that (P4) holds with constants $C_EC_X^3$ and $C_FC_Y^3$, respectively. \square

4. LOCAL RESULTS

In this section we show how to almost-diagonalize operators on finite dimensional mixed norm Hardy spaces and their duals, by building a block basis $(b_R^{(\varepsilon)})$ which satisfies the local product conditions (P1)–(P4). Moreover, if T has large diagonal with respect to the bi-parameter Haar system, then T has large diagonal with respect to the block basis $(b_R^{(\varepsilon)})$. This is achieved in Theorem 4.2.

Combining Theorem 4.2 with Theorem 3.1 yields the local factorization Theorem 4.6, which asserts that the identity operator on a finite dimensional mixed norm Hardy space (or its dual) factors through operators with large diagonal, in a larger, finite dimensional mixed norm Hardy space (or its dual).

As a by-product of Theorem 4.2 and Theorem 3.1, we obtain that the sequences of finite dimensional mixed norm Hardy spaces $(H_n^p(H_n^q))_{n \in \mathbb{N}}$ and $(H_n^p(H_n^q)^*)_{n \in \mathbb{N}}$ both have the property that projections almost annihilate finite dimensional subspaces, see Definition 5.2 and Theorem 4.4.

4.1. A combinatorial lemma in $H^p(H^q)$.

The following Lemma 4.1 will be used as a quantifiable substitute for weak limits in the proofs of the quantitative local results Theorem 4.2 and Theorem 4.6. Although the proof of Lemma 4.1 is merely a repetition of the proof given in [7] for the case $p = q = 1$, one still has to check that it does in fact work for the mixed norm Hardy spaces. For that reason and for sake of completeness, we give the proof below.

Lemma 4.1. *Let $1 \leq p, q < \infty$ and with the usual convention let $1 < p', q' \leq \infty$ denote the indices given by $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Let $i \in \mathbb{N}$, $K_0, L_0 \in \mathcal{D}$, and for all $0 \leq j \leq i-1$ let $x_j \in H^p(H^q)^*$ and $y_j \in H^p(H^q)$ be such that*

$$\sum_{j=0}^{i-1} \|x_j\|_{H^p(H^q)^*} \leq |K_0|^{1/p'} |L_0|^{1/q'} \quad \text{and} \quad \sum_{j=0}^{i-1} \|y_j\|_{H^p(H^q)} \leq |K_0|^{1/p} |L_0|^{1/q}. \quad (4.1)$$

The local frequency weight f_i is given by

$$f_i(K \times L) = \sum_{j=0}^{i-1} |\langle x_j, h_{K \times L} \rangle| + |\langle h_{K \times L}, y_j \rangle|, \quad K \times L \in \mathcal{R}. \quad (4.2)$$

Given $\tau > 0$, $r \in \mathbb{N}_0$, we define the collections of dyadic intervals

$$\begin{aligned} \mathcal{K}(K_0 \times L_0) &= \{K \times L_0 : K \subset K_0, |K| \leq 2^{-r}|K_0|, f_i(K \times L_0) \leq \tau|K \times L_0|\}, \\ \mathcal{L}(K_0 \times L_0) &= \{K_0 \times L : L \subset L_0, |L| \leq 2^{-r}|L_0|, f_i(K_0 \times L) \leq \tau|K_0 \times L|\}. \end{aligned}$$

For all integers k, ℓ the collections $\mathcal{K}_k(K_0 \times L_0)$ and $\mathcal{L}_\ell(K_0 \times L_0)$ are given by

$$\begin{aligned} \mathcal{K}_k(K_0 \times L_0) &= \mathcal{K}(K_0 \times L_0) \cap (\{K \in \mathcal{D} : |K| = 2^{-k}|K_0|\} \times \mathcal{D}), \\ \mathcal{L}_\ell(K_0 \times L_0) &= \mathcal{L}(K_0 \times L_0) \cap (\mathcal{D} \times \{L \in \mathcal{D} : |L| = 2^{-\ell}|L_0|\}). \end{aligned}$$

Let $\rho > 0$. Then there exist integers k, ℓ with

$$r \leq k, \ell \leq \left\lfloor \frac{4}{\rho^2 \tau^2} \right\rfloor + r \quad (4.3)$$

such that

$$|\mathcal{K}_k^*(K_0 \times L_0)| \geq (1 - \rho)|K_0 \times L_0| \quad \text{and} \quad |\mathcal{L}_\ell^*(K_0 \times L_0)| \geq (1 - \rho)|K_0 \times L_0|. \quad (4.4)$$

Note that the x -component of the rectangles in $\mathcal{K}_k(K_0 \times L_0)$ cover a set of measure $\geq (1 - \rho)|K_0|$ in K_0 , and the y -component of the rectangles in $\mathcal{L}_\ell(K_0 \times L_0)$ cover a set of measure $\geq (1 - \rho)|L_0|$ in L_0 . To be precise

$$\begin{aligned} \left| \bigcup \{K : K \times L \in \mathcal{K}_k(K_0 \times L_0)\} \right| &\geq (1 - \rho)|K_0|, \\ \left| \bigcup \{L : K \times L \in \mathcal{L}_\ell(K_0 \times L_0)\} \right| &\geq (1 - \rho)|L_0|. \end{aligned} \quad (4.5)$$

The estimate (4.5) follows from the fact that all $K \times L \in \mathcal{K}_k(K_0 \times L_0)$ are such that $L = L_0$, and similarly, the estimate (4.5) for the collection $\mathcal{L}_\ell(K_0 \times L_0)$ follow from the fact that all $K \times L \in \mathcal{L}_\ell(K_0 \times L_0)$ are such that $K = K_0$. See Figure 3 for a depiction of the collections $\mathcal{K}_k(K_0 \times L_0)$ and $\mathcal{L}_\ell(K_0 \times L_0)$.

Proof. Define $\mathcal{B} = \{K \times L_0 : K \subset K_0\} \setminus \mathcal{K}(K_0 \times L_0)$ and

$$\mathcal{B}_k = \mathcal{B} \cap (\{K \in \mathcal{D} : |K| = 2^{-k}|K_0|\} \times \mathcal{D}),$$

for all $k \in \mathbb{N}$. Now let

$$A = \left\lfloor \frac{4}{\rho^2 \tau^2} \right\rfloor + r.$$

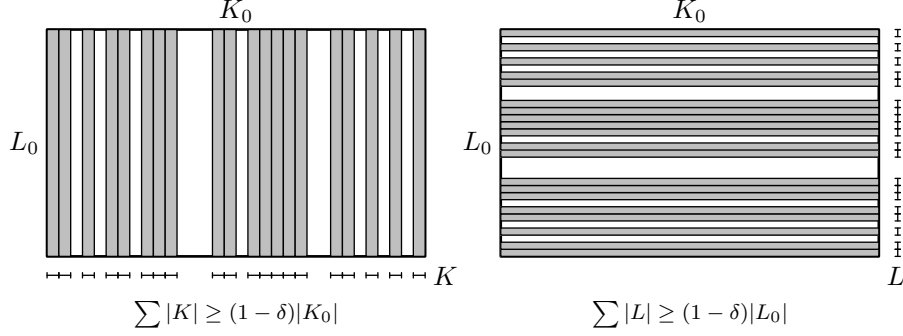


FIGURE 3. The gray rectangles in the left picture form the collection $\mathcal{K}_k(K_0 \times L_0)$, the gray rectangles to the right form $\mathcal{L}_\ell(K_0 \times L_0)$.

Since each $R \in \mathcal{B}(K_0 \times L_0)$ has the form $K \times L_0$ for some $K \in \mathcal{D}$, we have

$$\begin{aligned} \left\| \sum_{k=r}^A \sum_{R \in \mathcal{B}_k} \pm h_R \right\|_{H^p(H^q)} &= \left\| \sum_{k=r}^A \sum_{K: K \times L_0 \in \mathcal{B}_k} \pm h_K \right\|_{H^p} \|h_{L_0}\|_{H^q}, \\ \left\| \sum_{k=r}^A \sum_{R \in \mathcal{B}_k} \pm h_R \right\|_{H^p(H^q)^*} &= \left\| \sum_{k=r}^A \sum_{K: K \times L_0 \in \mathcal{B}_k} \pm h_K \right\|_{H^{p^*}} \|h_{L_0}\|_{H^{q^*}}. \end{aligned}$$

It is easily verified that $\|h_{L_0}\|_{H^q} = |L_0|^{1/q}$, $\|h_{L_0}\|_{H^{q^*}} = |L_0|^{1/q'}$, and that

$$\begin{aligned} \left\| \sum_{k=r}^A \sum_{K: K \times L_0 \in \mathcal{B}_k} \pm h_K \right\|_{H^p} &\leq \sqrt{A-r+1} |K_0|^{1/p}, \\ \left\| \sum_{k=r}^A \sum_{K: K \times L_0 \in \mathcal{B}_k} \pm h_K \right\|_{H^{p^*}} &\leq \sqrt{A-r+1} |K_0|^{1/p'}. \end{aligned}$$

Thus, we note the following estimates

$$\begin{aligned} \left\| \sum_{k=r}^A \sum_{R \in \mathcal{B}_k} \pm h_R \right\|_{H^p(H^q)} &\leq \sqrt{A-r+1} |K_0|^{1/p} |L_0|^{1/q}, \\ \left\| \sum_{k=r}^A \sum_{R \in \mathcal{B}_k} \pm h_R \right\|_{H^p(H^q)^*} &\leq \sqrt{A-r+1} |K_0|^{1/p'} |L_0|^{1/q'}. \end{aligned} \tag{4.6}$$

By construction, \mathcal{B}_k and $\mathcal{K}_k(K_0 \times L_0)$ form a disjoint decomposition of $K_0 \times L_0$. We will determine a collection $\mathcal{K}_k(K_0 \times L_0)$ by showing that \mathcal{B}_k^* is small enough for at least one value of k . Now assume the opposite, namely that

$$|\mathcal{B}_k^*| \geq \rho |K_0 \times L_0|, \quad r \leq k \leq A.$$

Summing these estimates yields

$$\sum_{k=r}^A |\mathcal{B}_k^*| \geq (A-r+1) \rho |K_0 \times L_0|, \tag{4.7}$$

Observe that by definition of \mathcal{B} and \mathcal{K}

$$\tau \cdot \sum_{k=r}^A |\mathcal{B}_k^*| \leq \sum_{j=0}^{i-1} \sum_{k=r}^A \sum_{K \times L_0 \in \mathcal{B}_k} |\langle x_j, h_{K \times L_0} \rangle| + |\langle h_{K \times L_0}, y_j \rangle|.$$

Rewriting the right hand side in the following way

$$\sum_{j=0}^{i-1} \left| \left\langle x_j, \sum_{k=r}^A \sum_{K \times L_0 \in \mathcal{B}_k} \pm h_{K \times L_0} \right\rangle \right| + \left| \left\langle \sum_{k=r}^A \sum_{K \times L_0 \in \mathcal{B}_k} \pm h_{K \times L_0}, y_j \right\rangle \right|,$$

and using (4.1) together with (4.6) yields

$$\tau \cdot \sum_{k=r}^A |\mathcal{B}_k^*| \leq 2\sqrt{A-r+1} |K_0 \times L_0|.$$

Combining the latter estimate with (4.7), we obtain

$$A \leq \frac{4}{\rho^2 \tau^2} + r - 1,$$

which contradicts the definition of A . Thus we found $r \leq k \leq A$ so that

$$|\mathcal{K}_k^*(K_0 \times L_0)| \geq (1 - \rho) |K_0 \times L_0|,$$

see Figure 3.

The same proof carried out in the other variable can be used to show the estimate for the collections $\mathcal{L}_\ell(K_0 \times L_0)$. \square

4.2. Quantitative almost-diagonalization.

We show that any given operator on a finite dimensional mixed normed Hardy space or its dual can be almost-diagonalized by a block basis $(b_R^{(\varepsilon)})$ of the bi-parameter Haar system. If moreover, the operator has large diagonal with respect to the bi-parameter Haar system, then it has large diagonal with respect to $(b_R^{(\varepsilon)})$. The block basis is such that it satisfies the local product conditions (P1)–(P4). We provide quantitative estimates on the number of block basis elements, which depends (among other things) on the dimension of the Hardy space.

Theorem 4.2. *Let $1 \leq p, q < \infty$ and $\delta \geq 0$. For $m, n \in \mathbb{N}$ and $\Gamma, \eta > 0$ there exists an integer $N = N(m, n, \Gamma, \eta)$ so that the following holds: For any operator $T : H_m^p(H_N^q) \rightarrow H_m^p(H_N^q)$ or $T : H_m^p(H_N^q)^* \rightarrow H_m^p(H_N^q)^*$ with $\|T\| \leq \Gamma$ satisfying*

$$\langle h_R, T h_R \rangle \geq \delta |R|, \quad R \in \mathcal{D}^m \times \mathcal{D}^N, \quad (4.8)$$

there exists a finite sequence of collections $(\mathcal{B}_R : R \in \mathcal{D}^m \times \mathcal{D}^n)$ and a sequence of signs $(\varepsilon_Q : Q \in \mathcal{R})$ defining a block basis of the Haar system $b_R^{(\varepsilon)}$ by

$$b_R^{(\varepsilon)} = \sum_{Q \in \mathcal{B}_R} \varepsilon_Q h_Q, \quad R \in \mathcal{D}^m \times \mathcal{D}^n,$$

so that the following conditions are satisfied:

- (i) $\mathcal{B}_R \subset \mathcal{D}^m \times \mathcal{D}^N$, for all $R \in \mathcal{D}^m \times \mathcal{D}^n$.
- (ii) $(\mathcal{B}_R : R \in \mathcal{D}^m \times \mathcal{D}^n)$ satisfies local product conditions (P1)–(P4) with constants $C_X = 1$ and $C_Y = 1 + \eta$.
- (iii) $(b_R^{(\varepsilon)} : R \in \mathcal{D}^m \times \mathcal{D}^n)$ almost-diagonalizes T so that T has large diagonal with respect to $(b_R^{(\varepsilon)} : R \in \mathcal{D}^m \times \mathcal{D}^n)$. To be more precise, we have the estimates

$$\sum_{\substack{R' \in \mathcal{D}^m \times \mathcal{D}^n \\ R' \neq R}} |\langle b_R^{(\varepsilon)}, T b_{R'}^{(\varepsilon)} \rangle| \leq \eta \|b_R^{(\varepsilon)}\|_2^2, \quad R \in \mathcal{D}^m \times \mathcal{D}^n, \quad (4.9a)$$

$$\langle b_R^{(\varepsilon)}, T b_R^{(\varepsilon)} \rangle \geq \delta \|b_R^{(\varepsilon)}\|_2^2, \quad R \in \mathcal{D}^m \times \mathcal{D}^n. \quad (4.9b)$$

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 FIGURE 4. Ordering of the 105 rectangles in $\mathcal{D}^2 \times \mathcal{D}^3$.

The proof of Theorem 4.2 relies on a modification of the construction in [7] for the sequence of collections of dyadic rectangles (\mathcal{B}_R) , the combinatorial Lemma 4.1 in $H^p(H^q)$ to make the off-diagonal small, and selecting signs (ε_Q) for the block basis $b_R^{(\varepsilon)} = \sum_{Q \in \mathcal{B}_R} \varepsilon_Q h_Q$ to keep the diagonal large. See [1] for the one-parameter setting.

Order relation \triangleleft on $\mathcal{D}^m \times \mathcal{D}^n$. The proof of Theorem 4.2 is by induction over the dyadic rectangles $\mathcal{D}^m \times \mathcal{D}^n$, hence, we need to linearly order them. To this end, let $<_\ell$ denote the lexicographic order on \mathbb{R}^4 . We define the linear order \triangleleft on $\mathcal{D}^m \times \mathcal{D}^n$ by

$$I_0 \times J_0 \triangleleft I_1 \times J_1 \quad \text{if and only if} \quad (|J_1|, |I_1|, \inf I_0, \inf J_0) <_\ell (|J_0|, |I_0|, \inf I_1, \inf J_1), \quad (4.10)$$

where $I_0 \times J_0, I_1 \times J_1 \in \mathcal{D}^m \times \mathcal{D}^n$. By $\mathcal{O}_{\triangleleft}$ we denote the index function given by the following conditions: The function

$$\mathcal{O}_{\triangleleft} : \mathcal{D}^m \times \mathcal{D}^n \rightarrow \{k \in \mathbb{Z} : 0 \leq k < (2^{m+1} - 1)(2^{n+1} - 1)\}$$

is bijective and satisfies

$$\mathcal{O}_{\triangleleft}(R_0) < \mathcal{O}_{\triangleleft}(R_1) \quad \text{if and only if} \quad R_0 \triangleleft R_1.$$

See Figure 4.

Proof of Theorem 4.2. We only proof the case where $T : H_m^p(H_N^q) \rightarrow H_m^p(H_N^q)$. The proof for the case $T : H_m^p(H_N^q)^* \rightarrow H_m^p(H_N^q)^*$ is the same, but the roles of T and T^* are reversed.

The proof is separated into the following steps:

- ▷ preparation,
- ▷ inductive construction of $b_{i_0}^{(\varepsilon)}$,
 - construction of \mathcal{B}_{i_0} ,

- selecting the signs ε ,
- ▷ verification that \mathcal{B}_{i_0} satisfies the local product conditions (P1)–(P4),
- ▷ verification that $b_{i_0}^{(\varepsilon)}$ almost-diagonalizes T .

Preparation. Let $1 \leq p, q < \infty$, $\delta \geq 0$, $m, n \in \mathbb{N}$ and $\Gamma, \eta > 0$. The number $N = N(m, n, \Gamma, \eta)$ will be determined in the course of the proof. Let $T : H_m^p(H_N^q) \rightarrow H_m^p(H_N^q)$ be an operator with $\|T\| \leq \Gamma$ such that

$$\langle h_Q, Th_Q \rangle \geq \delta |Q|, \quad Q \in \mathcal{D}^m \times \mathcal{D}^N. \quad (4.11)$$

Given $Q \in \mathcal{D}^m \times \mathcal{D}^N$ we write

$$Th_Q = \alpha_Q h_Q + r_Q, \quad (4.12a)$$

where

$$\alpha_Q = \frac{\langle h_Q, Th_Q \rangle}{|Q|} \quad \text{and} \quad r_Q = \sum_{Q' : Q' \neq Q} \frac{\langle h_{Q'}, Th_{Q'} \rangle}{|Q'|} h_{Q'}. \quad (4.12b)$$

Note that for all $Q = K \times L \in \mathcal{D}^m \times \mathcal{D}^N$ we have the estimates

$$\delta \leq \alpha_Q \leq \|T\| \quad \text{and} \quad \|r_Q\|_{H^p(H^q)} \leq 2\|T\| |K|^{1/p} |L|^{1/q}. \quad (4.13)$$

Inductive construction of $(b_R^{(\varepsilon)} : R \in \mathcal{D}^m \times \mathcal{D}^N)$. For fixed $R \in \mathcal{D}^m \times \mathcal{D}^N$, the block basis element $b_R^{(\varepsilon)}$ will be determined by a collection of dyadic rectangles $\mathcal{B}_R \subset \mathcal{D}^m \times \mathcal{D}^N$ and signs $\varepsilon = (\varepsilon_Q : Q \in \mathcal{D}^m \times \mathcal{D}^N)$, and is of the following form:

$$b_R^{(\varepsilon)} = \sum_{Q \in \mathcal{B}_R} \varepsilon_Q h_Q. \quad (4.14)$$

From now on, we systematically use the following rule: whenever $\mathcal{O}_{\triangleleft}(R) = i$, we set

$$\mathcal{B}_i = \mathcal{B}_R \quad \text{and} \quad b_i^{(\varepsilon)} = b_R^{(\varepsilon)}.$$

In the course of this proof we will construct the finite sequence of collections $(\mathcal{B}_R : R \in \mathcal{D}^m \times \mathcal{D}^N)$ and signs $\varepsilon = (\varepsilon_Q : Q \in \mathcal{D}^m \times \mathcal{D}^N)$ so that $(\mathcal{B}_R : R \in \mathcal{D}^m \times \mathcal{D}^N)$ satisfies the local product conditions (P1)–(P4) with constants $C_X = 1$ and $C_Y = 1 + \eta$, and that the block basis $(b_i^{(\varepsilon)})_{i \in \mathbb{N}_0}$ given by (4.14) satisfies

$$\sum_{j=0}^{i-1} |\langle T^* b_j^{(\varepsilon)}, b_i^{(\varepsilon)} \rangle| + |\langle b_i^{(\varepsilon)}, T b_j^{(\varepsilon)} \rangle| \leq c(\eta') 4^{-i-1} \|b_i^{(\varepsilon)}\|_2^2, \quad (4.15a)$$

$$\langle b_i^{(\varepsilon)}, T b_i^{(\varepsilon)} \rangle \geq \delta \|b_i^{(\varepsilon)}\|_2^2, \quad (4.15b)$$

for all $i \in \mathbb{N}_0$, where $c(\eta') \rightarrow 0$ as $\eta' \rightarrow 0$. We now choose $\eta' = \eta'(m, n, \Gamma, \eta) > 0$ so small that

$$(1 - \eta')^{-1} \leq 1 + \eta \quad \text{and} \quad \eta' (1 - \eta')^{-2} 4^{m+n} \Gamma \leq \eta. \quad (4.16)$$

The induction begins by putting

$$\mathcal{B}_0 = \{[0, 1) \times [0, 1)\} \quad \text{and} \quad b_0^{(\varepsilon)} = h_{[0, 1) \times [0, 1)}. \quad (4.17)$$

Let $i_0 \in \mathbb{N}$. At this stage we assume:

- ▷ There exist collections $\mathcal{B}_j = \mathcal{B}_{I \times J}$, for all $\mathcal{O}_{\triangleleft}(I \times J) = j \leq i_0 - 1$ of the form

$$\mathcal{B}_{I \times J} = \{I \times L : L \in \mathcal{Y}_{I \times J}\},$$

where $\mathcal{Y}_{I \times J}$ is a finite subset of \mathcal{D} . In the notation of Section 3, $\mathcal{X}_{I \times J} = \{I\}$.

- ▷ The collections \mathcal{B}_j , $0 \leq j \leq i_0 - 1$, are such that $(\mathcal{B}_j)_{j=0}^{i_0-1}$ satisfies the local product conditions (P1)–(P4).

- ▷ The block basis elements $b_j^{(\varepsilon)}$, given by (4.14), satisfy (4.15) for $0 \leq j \leq i_0 - 1$.

Now, we turn to the construction of \mathcal{B}_{i_0} and ε_Q , where $Q \in \mathcal{B}_{i_0}$. In the first step we find \mathcal{B}_{i_0} , and only then we will determine the signs ε .

Construction of \mathcal{B}_{i_0} . Let $I_0 \times J_0 \in \mathcal{R}$ such that $\mathcal{O}_{\triangleleft}(I_0 \times J_0) = i_0$. At the beginning of the construction as well as at the end, we will distinguish between the two cases

$$I_0 \neq [0, 1) \quad \text{and} \quad I_0 = [0, 1).$$

In both cases, we will use the combinatorial Lemma 4.1.

If $I_0 \neq [0, 1)$ we define the collection $\mathcal{P}_{I_0 \times J_0}$ by

$$\mathcal{P}_{I_0 \times J_0} = \{I \times J \in \mathcal{D}^m \times \mathcal{D}^n : I \times J \triangleleft I_0 \times J_0, I \neq I_0\}, \quad (4.18a)$$

and if $I_0 = [0, 1)$, we put

$$\mathcal{P}_{[0,1) \times J_0} = \{I \times J \in \mathcal{D}^m \times \mathcal{D}^n : I \times J \triangleleft [0, 1) \times J_0, |J| > |J_0|\}. \quad (4.18b)$$

In both cases, we now define $\mathbb{A}_{I_0 \times J_0}$ by

$$\mathbb{A}_{I_0 \times J_0} = \{\{I \times J' \in \mathcal{P}_{I_0 \times J_0} : |J'| = |J|\} : I \times J \in \mathcal{P}_{I_0 \times J_0}\}. \quad (4.19)$$

Before we proceed with the proof, we make a few remarks.

- ▷ For all $J \in \mathcal{D}^n$ with $|J| = |J_0|$ holds that $\mathcal{P}_{I_0 \times J} = \mathcal{P}_{I_0 \times J_0}$, and hence $\mathbb{A}_{I_0 \times J} = \mathbb{A}_{I_0 \times J_0}$.
- ▷ If $I \times J \in \mathcal{A} \in \mathbb{A}_{I_0 \times J_0}$, then

$$\mathcal{A} = \{I \times J' : J' \in \mathcal{D}, |J'| = |J|\} \quad (4.20)$$

see (4.10) and (4.19).

- ▷ The collection $\mathbb{A}_{I_0 \times J_0}$ is a partition of $\mathcal{P}_{I_0 \times J_0}$, i.e.

$$\bigcup \mathbb{A}_{I_0 \times J_0} = \mathcal{P}_{I_0 \times J_0} \quad \text{and} \quad \mathcal{A} \cap \mathcal{A}' = \emptyset$$

for all $\mathcal{A}, \mathcal{A}' \in \mathbb{A}_{I_0 \times J_0}$ with $\mathcal{A} \neq \mathcal{A}'$.

- ▷ The collections $\mathcal{B}_{I \times J}$ have already been constructed for all $I \times J \in \mathcal{P}_{I_0 \times J_0}$.
- ▷ Let $\mathcal{A} \in \mathbb{A}_{I_0 \times J_0}$ and $I \times J, I \times J' \in \mathcal{A}$, then

$$Y_{I \times J} \cap Y_{I \times J'} = \emptyset, \quad \text{if } J \neq J'.$$

For each $\mathcal{A} \in \mathbb{A}_{I_0 \times J_0}$, let $W(\mathcal{A})$ denote the set given by

$$W(\mathcal{A}) = \bigcup_{I \times J \in \mathcal{A}} Y_{I \times J}. \quad (4.21)$$

Note that for each $\mathcal{A} \in \mathbb{A}_{I_0 \times J_0}$ the set $W(\mathcal{A})$ is an almost cover of the unit interval. Now we define $W_{I_0 \times J_0}$ by intersecting all the $W(\mathcal{A})$:

$$W_{I_0 \times J_0} = \bigcap_{\mathcal{A} \in \mathbb{A}_{I_0 \times J_0}} W(\mathcal{A}). \quad (4.22)$$

We will cover the set $W_{I_0 \times J_0}$ with smaller intervals than we have previously used in our construction. To this end let

$$\gamma_{i_0} = \gamma_{I_0 \times J_0} = \frac{1}{2} \min\{|L| : \exists I \times J \triangleleft I_0 \times J_0, \mathcal{B}_{I \times J} \ni L\} \quad (4.23)$$

and define the high frequency cover $\mathcal{W}_{I_0 \times J_0}$ of $W_{I_0 \times J_0}$ by

$$\mathcal{W}_{I_0 \times J_0} = \{L_0 \in \mathcal{D} : |L_0| = \gamma_{I_0 \times J_0}, L_0 \subset W_{I_0 \times J_0}\}. \quad (4.24)$$

Note the following identity:

$$\bigcup \mathcal{W}_{I_0 \times J_0} = W_{I_0 \times J_0}. \quad (4.25)$$

To each of the rectangles $I_0 \times L_0$, where $L_0 \in \mathcal{W}_{I_0 \times J_0}$, we will now prepare to apply Lemma 4.1, so that I_0 will remain intact, and L_0 will be almost covered with high frequencies L . To this end let

$$\beta_{i_0} = \beta_{I_0 \times J_0} = \min\{|I_0 \times L_0| : L_0 \in \mathcal{W}_{I_0 \times J_0}\} \quad (4.26)$$

and define for all $0 \leq j \leq i_0 - 1$

$$x_j := \frac{\beta_{i_0}}{\Gamma i_0 \|b_j^{(\varepsilon)}\|_{H^p(H^q)^*}} T^* b_j^{(\varepsilon)}, \quad y_j := \frac{\beta_{i_0}}{\Gamma i_0 \|b_j^{(\varepsilon)}\|_{H^p(H^q)}} T b_j^{(\varepsilon)}. \quad (4.27)$$

Recall that $\|T\| \leq \Gamma$, hence

$$\sum_{j=0}^{i_0-1} \|x_j\|_{H^p(H^q)^*} \leq \beta_{i_0} \quad \text{and} \quad \sum_{j=0}^{i_0-1} \|y_j\|_{H^p(H^q)} \leq \beta_{i_0}.$$

The local frequency weight f_{i_0} is given by

$$f_{i_0}(Q) = \sum_{j=0}^{i_0-1} |\langle x_j, h_Q \rangle| + |\langle y_j, h_Q \rangle|, \quad Q \in \mathcal{R}, \quad (4.28)$$

and the constant τ_{i_0} by

$$\tau_{i_0} = \tau_{I_0 \times J_0} = \frac{\eta' \beta_{i_0}}{i_0 4^{i_0+1}}. \quad (4.29)$$

For each $L_0 \in \mathcal{W}_{I_0 \times J_0}$ we put

$$\mathcal{L}(I_0 \times L_0) = \{I_0 \times L : L \in \mathcal{D}, L \subsetneq L_0, f_{i_0}(I_0 \times L) \leq \tau_{i_0} |I_0 \times L|\}. \quad (4.30)$$

Finally, we define the constant ρ_{i_0} by

$$\rho_{i_0} = \rho_{I_0 \times J_0} = \frac{\eta'}{4^{i_0}}. \quad (4.31)$$

Since $\beta_{i_0} \leq |I_0|^{1/p'} |L_0|^{1/q'}$ and $\beta_{i_0} \leq |I_0|^{1/p} |L_0|^{1/q}$, for all $L_0 \in \mathcal{W}_{I_0 \times J_0}$, Lemma 4.1 yields an integer $\ell = \ell(I_0 \times L_0)$ with

$$1 \leq \ell \leq \left\lfloor \frac{4}{\rho_{i_0}^2 \tau_{i_0}^2} \right\rfloor + 1, \quad (4.32)$$

such that the collection $\mathcal{L}_\ell(I_0 \times L_0)$ given by

$$\mathcal{L}_\ell(I_0 \times L_0) = \{I_0 \times L \in \mathcal{L}(I_0 \times L_0) : |L| = 2^{-\ell} |L_0|\}$$

satisfies the estimate

$$(1 - \rho_{i_0}) |I_0 \times L_0| \leq \left| \bigcup \{Q : Q \in \mathcal{L}_\ell(I_0 \times L_0)\} \right| \leq |I_0 \times L_0|.$$

Now, we take the union over all $L_0 \in \mathcal{W}_{I_0 \times J_0}$ to obtain

$$\mathcal{Z}_{I_0 \times J_0} = \bigcup \{\mathcal{L}_\ell(I_0 \times L_0) : L_0 \in \mathcal{W}_{I_0 \times J_0}\}. \quad (4.33)$$

Once again, we emphasize that $\ell = \ell(I_0 \times L_0)$ in the above formula. Let $Z_{I_0 \times J_0}$ denote the pointset of $\mathcal{Z}_{I_0 \times J_0}$, i.e.

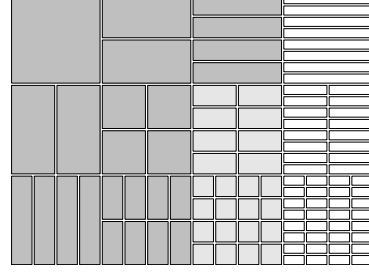
$$Z_{I_0 \times J_0} = \bigcup \mathcal{Z}_{I_0 \times J_0},$$

then for all $L_0 \in \mathcal{W}_{I_0 \times J_0}$ we have the estimates

$$(1 - \rho_{i_0}) |I_0 \times L_0| \leq \left| Z_{I_0 \times J_0} \cap (I_0 \times L_0) \right| \leq |I_0 \times L_0|. \quad (4.34)$$

We want to point out that $Q \in \mathcal{Z}_{I_0 \times J_0}$ implies $Q = I_0 \times L$, for some $L \in \mathcal{D}$. There exists a unique $L_0 \in \mathcal{W}_{I_0 \times J_0}$ such that $L_0 \supset L$, and therefore $|L| = 2^{-\ell}$, where $\ell = \ell(I_0 \times L_0)$.

Case 1: $I_0 \neq [0, 1)$. Here, we know that $2^{-m} \leq |I_0| \leq 1/2$. Let \tilde{I}_0 be the dyadic predecessor of I_0 , then $\mathcal{B}_{\tilde{I}_0 \times J_0}$ has already been defined (see (4.10)). The block basis indexed by the dark gray rectangles has already been constructed. Here, we determine the block basis for the light gray rectangles. The white ones will be treated later.



In this case we put

$$\mathcal{B}_{I_0 \times J_0} = \{I_0 \times L \in \mathcal{Z}_{I_0 \times J_0} : I_0 \times L \subset B_{\tilde{I}_0 \times J_0}\}. \quad (4.35)$$

see Figure 5.

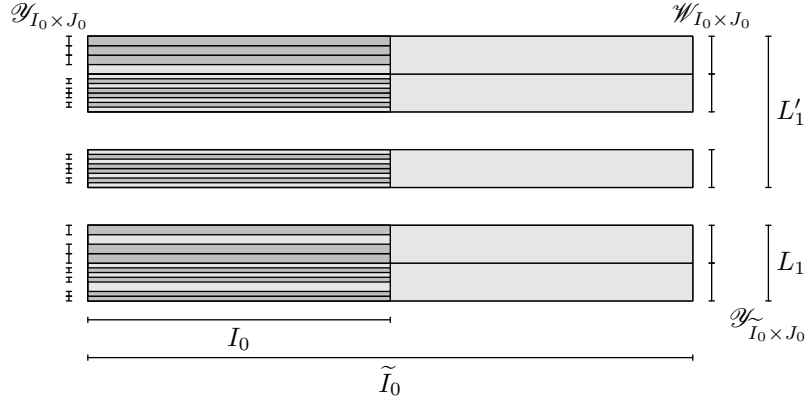


FIGURE 5. In this figure, the dyadic interval I_0 is the left half of \tilde{I}_0 , and the collection $\mathcal{Y}_{I_0 \times J_0}$ consists of the dyadic intervals L_1 and L'_1 . Immediately to the left of L_1 , L'_1 are the unlabeled dyadic intervals of the collection $\mathcal{W}_{I_0 \times J_0}$. Note that all the intervals in $\mathcal{W}_{I_0 \times J_0}$ have the same length, see (4.24). The pointset $W_{I_0 \times J_0}$ of $\mathcal{W}_{I_0 \times J_0}$ almost covers each of the intervals L_1 , L'_1 , see (4.19), (4.20), (4.21), (4.22) and (4.25). The light gray rectangles are formed by the product of \tilde{I}_0 with the intervals in $\mathcal{W}_{I_0 \times J_0}$. The collection of dyadic intervals $\mathcal{Y}_{I_0 \times J_0}$ is given by the unlabeled vertical intervals to the very left of the figure are determined by the combinatorial Lemma 4.1. The pointset $Y_{I_0 \times J_0}$ of $\mathcal{Y}_{I_0 \times J_0}$ almost covers each interval in $\mathcal{W}_{I_0 \times J_0}$, and therefore each interval in $\mathcal{Y}_{I_0 \times J_0} = \{L_1, L'_1\}$, see (4.30), (4.33) (4.34) and (4.35). The collection $\mathcal{B}_{I_0 \times J_0}$ consists of the dark gray rectangles and is given by $\{I_0 \times L : L \in \mathcal{Y}_{I_0 \times J_0}\}$.

By (4.35) we obtain that

$$K \times L \in \mathcal{B}_{I_0 \times J_0} \quad \text{implies} \quad K = I_0, \quad (4.36)$$

thus, (4.34), (4.35) and (4.36) yield

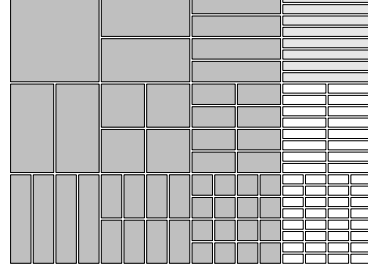
$$\mathcal{X}_{I_0 \times J_0} = \{I_0\} \quad (4.37a)$$

and

$$(1 - \rho_{I_0 \times J_0})|L| \leq |Y_{I_0 \times J_0} \cap L| \leq |L|, \quad (4.37b)$$

for all $L \in \mathcal{Y}_{I \times J}$ with $L \cap Y_{I_0 \times J_0} \neq \emptyset$, where $I \times J \in \mathcal{P}_{I_0 \times J_0}$ is maximal with respect to the ordering \triangleleft (in which case $I \neq I_0$, $J = J_0$ and we have $L \cap Y_{I_0 \times J_0} \neq \emptyset$ for all $L \in \mathcal{Y}_{I \times J}$).

Case 2: $I_0 = [0, 1)$. In this case we know that $\mathcal{B}_{I \times \tilde{J}_0}$ has already been constructed for all $I \in \mathcal{D}^m$ (see (4.10)); those are the dark gray rectangles in the third column. Here, we determine the block basis for the light gray rectangles. The white ones will be treated later.



Here, we define the sets

$$B_{[0,1) \times \tilde{J}_0}^\ell = \bigcup_{[0,1) \times L_0 \in \mathcal{B}_{[0,1) \times \tilde{J}_0}} [0, 1) \times L_0^\ell$$

and

$$B_{[0,1) \times \tilde{J}_0}^r = \bigcup_{[0,1) \times L_0 \in \mathcal{B}_{[0,1) \times \tilde{J}_0}} [0, 1) \times L_0^r.$$

If J_0 is the *left half* of \tilde{J}_0 we put

$$\mathcal{B}_{[0,1) \times J_0} = \{[0, 1) \times L \in \mathcal{Z}_{[0,1) \times J_0} : [0, 1) \times L \subset B_{[0,1) \times \tilde{J}_0}^\ell\}. \quad (4.38a)$$

If J_0 is the *right half* of \tilde{J}_0 we put

$$\mathcal{B}_{[0,1) \times J_0} = \{[0, 1) \times L \in \mathcal{Z}_{[0,1) \times J_0} : [0, 1) \times L \subset B_{[0,1) \times \tilde{J}_0}^r\}, \quad (4.38b)$$

see Figure 6.

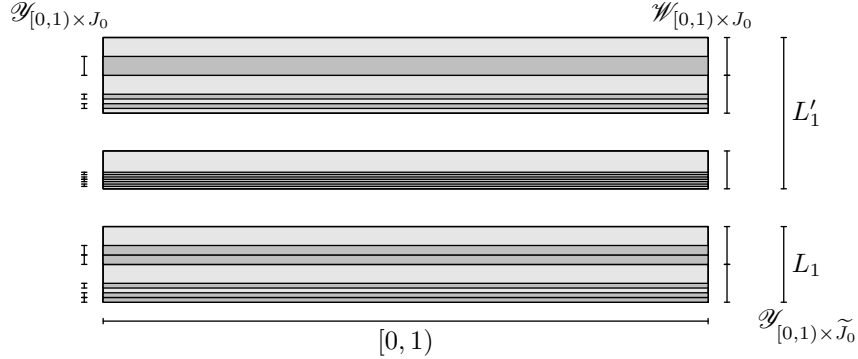


FIGURE 6. In this figure, J_0 is the left half of \tilde{J}_0 . The collection of dyadic intervals $\mathcal{Y}_{[0,1) \times \tilde{J}_0} = \{L_1, L_1'\}$ are the y -components of the collection $\mathcal{B}_{[0,1) \times \tilde{J}_0}$. To the left is the collection $\mathcal{W}_{[0,1) \times J_0}$, whose pointset $W_{[0,1) \times J_0}$ almost covers L_1 and L_1' . The light gray dyadic rectangles are determined by the products $[0, 1) \times L_0$, where $L_0 \in \mathcal{W}_{[0,1) \times J_0}$. The dark gray rectangles are the output of the combinatorial Lemma 4.1, and are all contained in the left halves in one of the intervals $L_0 \in \mathcal{W}_{[0,1) \times J_0}$ (since J_0 is the left half of \tilde{J}_0 , see (4.38)). The dark gray rectangles form the collection $\mathcal{B}_{[0,1) \times J_0}$. The collection $\mathcal{Y}_{[0,1) \times J_0}$ is the collection of y -components of $\mathcal{B}_{[0,1) \times J_0}$.

In this case, we have by (4.38) that

$$K \times L \in \mathcal{B}_{[0,1) \times J_0} \quad \text{implies} \quad K = [0, 1), \quad (4.39)$$

thus, (4.34), (4.38) and (4.39) yield

$$\mathcal{X}_{[0,1) \times J_0} = \{[0, 1)\} \quad (4.40a)$$

and

$$\frac{1}{2}(1 - 2\rho_{[0,1) \times J_0})|L| \leq |Y_{[0,1) \times J_0} \cap L| \leq |L|, \quad (4.40b)$$

for all $L \in \mathcal{Y}_{I \times J}$ with $L \cap Y_{I_0 \times J_0} \neq \emptyset$, where $I \times J \in \mathcal{P}_{I_0 \times J_0}$ is maximal with respect to the ordering \triangleleft (in which case $I = [1 - 2^{-m}, 1)$, $J = \tilde{J}_0$ and $L \cap Y_{I_0 \times J_0} \neq \emptyset$ for all $L \in \mathcal{Y}_{I \times J}$).

Recall that $i_0 = \mathcal{O}_{\triangleleft}(I_0 \times J_0)$, and that $\mathcal{B}_{i_0} = \mathcal{B}_{I_0 \times J_0}$ as well as $b_{i_0}^{(\varepsilon)} = b_{I_0 \times J_0}^{(\varepsilon)}$. By the definition of $\mathcal{L}(I_0 \times L_0)$ (see (4.30)) and the definition of \mathcal{B}_{i_0} (see (4.35) and (4.38)), we note that in both cases

$$f_{i_0}(Q) \leq \tau_{i_0}|Q|, \quad Q \in \mathcal{B}_{i_0}. \quad (4.41)$$

Selecting the signs ε . In any of the above cases (4.35) and (4.38), we define the following function. For any choice of signs $\varepsilon_Q \in \{-1, +1\}$, $Q \in \mathcal{B}_{i_0}$ put

$$b_{i_0}^{(\varepsilon)} = \sum_{Q \in \mathcal{B}_{i_0}} \varepsilon_Q h_Q. \quad (4.42)$$

We will now select signs $(\varepsilon_Q : Q \in \mathcal{B}_{i_0})$ such that

$$\langle b_{i_0}^{(\varepsilon)}, T b_{i_0}^{(\varepsilon)} \rangle \geq \delta \|b_{i_0}^{(\varepsilon)}\|_2^2.$$

To this end observe that (4.12) and (4.42) yields

$$\langle b_{i_0}^{(\varepsilon)}, T b_{i_0}^{(\varepsilon)} \rangle = \sum_{Q \in \mathcal{B}_{i_0}} \alpha_Q |Q| + \langle b_{i_0}^{(\varepsilon)}, R_m^{(\varepsilon)} \rangle,$$

where

$$R_m^{(\varepsilon)} = \sum_{Q \in \mathcal{B}_{i_0}} \varepsilon_Q r_Q.$$

Thus, by (4.13) we get

$$\langle b_{i_0}^{(\varepsilon)}, T b_{i_0}^{(\varepsilon)} \rangle \geq \delta \|b_{i_0}^{(\varepsilon)}\|_2^2 + \langle b_{i_0}^{(\varepsilon)}, R_m^{(\varepsilon)} \rangle,$$

Let \mathbb{E}_ε denote the average over all possible choices of signs $(\varepsilon_Q : Q \in \mathcal{B}_{i_0})$. Using $\langle h_Q, r_Q \rangle = 0$, we obtain

$$\langle b_{i_0}^{(\varepsilon)}, R_m^{(\varepsilon)} \rangle = \sum \varepsilon_{Q_0} \varepsilon_{Q_1} \langle h_{Q_0}, r_{Q_1} \rangle,$$

where the sum is taken over all $Q_0, Q_1 \in \mathcal{B}_{i_0}$ with $Q_0 \neq Q_1$. Taking the expectation on the right hand side we obtain,

$$\mathbb{E}_\varepsilon \langle b_{i_0}^{(\varepsilon)}, R_m^{(\varepsilon)} \rangle = 0.$$

This gives

$$\mathbb{E}_\varepsilon \langle b_{i_0}^{(\varepsilon)}, T b_{i_0}^{(\varepsilon)} \rangle \geq \delta \mathbb{E}_\varepsilon \|b_{i_0}^{(\varepsilon)}\|_2^2, \quad (4.43)$$

hence there exists at least one ε such that

$$\langle b_{i_0}^{(\varepsilon)}, T b_{i_0}^{(\varepsilon)} \rangle \geq \delta \|b_{i_0}^{(\varepsilon)}\|_2^2. \quad (4.44)$$

$(\mathcal{B}_i : i \leq i_0)$ satisfies the local product conditions (P1)–(P4). It should be clear from the definition of \mathcal{F}_m in each step, that $(\mathcal{B}_i : i \leq i_0)$ satisfies (P1). Since $\mathcal{X}_{I \times J} = \{I\}$ for all $I \times J \in \mathcal{D}^m \times \mathcal{D}^n$, (P2)–(P4) is satisfied with $C_X = 1$. Recalling the definition of Y_J (see (3.4)), and that the new y -components are obtained by intersecting all the supports from the previous steps (see (4.21), (4.22), (4.24), (4.25), (4.30), (4.33), (4.35), and (4.38)) we observe that

$$Y_J = \bigcup_{I \in \mathcal{D}^m} Y_{I \times J} = Y_{[0,1] \times J}, \quad J \in \mathcal{D}^n.$$

By considering (4.38) together with the above identity, it should be clear that $(Y_J : J \in \mathcal{D}^n)$ satisfies (P2) and $|Y_J| \leq |J|$, $J \in \mathcal{D}^n$. The remaining measure estimates (P3) and (P4) follow by induction from (4.37) and (4.40).

Now, let $I_0 \times J_0, I \times J \in \mathcal{D}^m \times \mathcal{D}^n$ with $I_0 \times J_0 \subsetneq I \times J$ and let $L \in \mathcal{Y}_{I \times J}$. From (4.37) and (4.40) follows immediately that $K_0 = I_0$ and $K = I$. Since the ρ_i are a geometric sequence (see (4.31)), we obtain by induction that

$$\frac{|J_0|}{|J|}(1 - \eta')|L| \leq |L \cap Y_{I_0 \times J_0}| \leq \frac{|J_0|}{|J|}|L|. \quad (4.45)$$

We remark that (4.45) implies (P3) and (P4) with $C_Y = C_Y(\eta') = (1 - \eta')^{-1}$. To summarize, we showed that $(\mathcal{B}_R : R \in \mathcal{D}^m \times \mathcal{D}^n)$ satisfies the local product conditions (P1)–(P4) with constants $C_X = 1$ and $C_Y = (1 - \eta')^{-1}$.

The block basis $(b_i^{(\epsilon)} : i \leq i_0)$ almost-diagonalizes T . First, recall that the constant τ_i was given by $\tau_i = \frac{\eta' \beta_i}{i 4^{i+1}}$ (see (4.26) and (4.29)). With that in mind, we collect the estimates (4.41), (see (4.28) for the definition of f_{i_0}) and the mixed norm estimates in Theorem 3.1 to obtain

$$\sum_{j=0}^{i-1} |\langle T^* b_j^{(\epsilon)}, h_Q \rangle| + |\langle h_Q, T b_j^{(\epsilon)} \rangle| \leq \eta'(1 - \eta')^{-1} 2^{m+n} \Gamma 4^{-i-1} |Q|,$$

for all i and $Q \in \mathcal{B}_i$. From the latter estimate and the definition of $b_i^{(\epsilon)}$ (see (4.42)) we obtain by summing over $Q \in \mathcal{B}_i$

$$\sum_{j=0}^{i-1} |\langle T^* b_j^{(\epsilon)}, b_i^{(\epsilon)} \rangle| + |\langle b_i^{(\epsilon)}, T b_j^{(\epsilon)} \rangle| \leq \eta'(1 - \eta')^{-1} 2^{m+n} \Gamma 4^{-i-1}, \quad i \geq 1. \quad (4.46)$$

From the first term in the sum of (4.46) follows the estimate

$$|\langle b_i^{(\epsilon)}, T b_j^{(\epsilon)} \rangle| \leq \eta'(1 - \eta')^{-1} 2^{m+n} \Gamma 4^{-j-1}, \quad j > i \geq 0.$$

Summing over all those j we obtain

$$\sum_{j \geq i+1} |\langle b_i^{(\epsilon)}, T b_j^{(\epsilon)} \rangle| \leq \eta'(1 - \eta')^{-1} 2^{m+n} \Gamma 4^{-i-1}, \quad i \geq 0.$$

Combining the latter estimate with (4.46) and using that $\|b_i\|_2^2 \geq (1 - \eta') 2^{-m-n}$ (see (3.7) and recall that $b_i = b_R$, whenever $i = \mathcal{O}_{\triangleleft}(R)$, and that here $R \in \mathcal{D}^m \times \mathcal{D}^n$) yields

$$\sum_{j:j \neq i} |\langle b_i^{(\epsilon)}, T b_j^{(\epsilon)} \rangle| \leq \eta'(1 - \eta')^{-2} 4^{m+n} \Gamma \|b_i^{(\epsilon)}\|_2^2, \quad i \geq 0. \quad (4.47)$$

We remark that (4.47) and (4.16) together with (4.44) proves (4.9).

Finally, observe that (i) of Theorem 4.2 holds true by observing that all the constants in the proof depend only on m, n, Γ and η (see (4.16), (4.23), (4.26), (4.29), (4.30), (4.31) and (4.32)). \square

Remark 4.3. Let (e_K) and (f_L) denote block bases of the one parameter Haar system satisfying the hypothesis of the reiteration Lemma 3.2, that is $(e_K \otimes f_L)$ satisfies the local product conditions (P1)–(P4), and additional regularity assumptions (see Lemma 3.2 (i) and (ii)).

We remark that we could repeat the proof of Theorem 4.2 with $h_{K \times L}$ replaced by $\tilde{h}_{K \times L} = e_K \otimes f_L$, $K \times L \in \mathcal{D}^m \times \mathcal{D}^n$. Due to the reiteration Lemma 3.2, we would arrive at the same conclusion. To be more precise: if we replace (4.8) in Theorem 4.2 by

$$\langle \tilde{h}_Q, T\tilde{h}_Q \rangle \geq \delta \|\tilde{h}_Q\|_2^2, \quad Q \in \mathcal{D}^m \times \mathcal{D}^n, \quad (4.48)$$

then all the conclusions (i)–(iii) of Theorem 4.2 remain valid for the block basis $(\tilde{b}_R^{(\varepsilon)} : R \in \mathcal{D}^m \times \mathcal{D}^n)$ of the bi-parameter Haar system given by

$$\tilde{b}_R^{(\varepsilon)} = \sum_{Q \in \mathcal{B}_R} \varepsilon_Q \tilde{h}_Q, \quad R \in \mathcal{D}^m \times \mathcal{D}^n.$$

4.3. Projections that almost annihilate finite dimensional subspaces.

In the proof of the main result Theorem 2.1, we will use the almost-diagonalization result Theorem 4.2. Additionally, we will need the following variation of Theorem 4.2.

Theorem 4.4. *Let $1 \leq p, q < \infty$, $m, n, d \in \mathbb{N}$ and $\eta > 0$. Then there exists an integer $N = N(m, n, d, \eta)$ so that for any d -dimensional subspace $F \subset H_m^p(H_N^q)$ (respectively $F \subset H_m^p(H_N^q)^*$) there exists a block basis $(b_R : R \in \mathcal{D}^m \times \mathcal{D}^n)$ satisfying the following conditions:*

- (i) $\mathcal{B}_R \subset \mathcal{D}^m \times \mathcal{D}^n$, for all $R \in \mathcal{D}^m \times \mathcal{D}^n$.
- (ii) For every finite sequence of scalars $(a_R : R \in \mathcal{D}^m \times \mathcal{D}^n)$ we have that

$$(1 + \eta)^{-1} \left\| \sum_{R \in \mathcal{D}^m \times \mathcal{D}^n} a_R h_R \right\| \leq \left\| \sum_{R \in \mathcal{D}^m \times \mathcal{D}^n} a_R b_R \right\| \leq (1 + \eta) \left\| \sum_{R \in \mathcal{D}^m \times \mathcal{D}^n} a_R h_R \right\|. \quad (4.49)$$

The above norms are either all the norm of $H^p(H^q)$, or they are all the norm of $H^p(H^q)^*$.

- (iii) The orthogonal projection $Q : H_m^p(H_N^q) \rightarrow H_m^p(H_N^q)$ (respectively $Q : H_m^p(H_N^q)^* \rightarrow H_m^p(H_N^q)^*$) given by

$$Qf = \sum_{R \in \mathcal{D}^m \times \mathcal{D}^n} \frac{\langle f, b_R \rangle}{\|b_R\|_2^2} b_R$$

satisfies the estimates

$$\begin{aligned} \|Qf\| &\leq (1 + \eta) \|f\|, \quad f \in H_m^p(H_N^q) \text{ (respectively } f \in H_m^p(H_N^q)^*), \\ \|Qf\| &\leq \eta \|f\|, \quad f \in F. \end{aligned} \quad (4.50)$$

The above norms are either all the norm of $H^p(H^q)$, or they are all the norm of $H^p(H^q)^*$.

Proof. The proof of Theorem 4.4 is a repetition of the almost-diagonalization argument in the proof of Theorem 4.2, where the combinatorial Lemma 4.1 is used with the following frequency weight f in each step. Given a finite $\eta/2$ -net $(y_j)_{j=1}^k$ of the unit ball in F define the local frequency weight f by

$$f(R) = \sum_{j=1}^k |\langle h_R, y_j \rangle|, \quad R \in \mathcal{R}.$$

Since we do not need a large diagonal in this particular instance, we choose all signs $\varepsilon_Q = 1$. The bi-parameter case is analogous to the one parameter case, which is described in detail in [11, 290–291]. \square

Remark 4.5. In view of Remark 3.3 and Remark 4.3, it is clear that we could have replaced the bi-parameter Haar system $(h_K \otimes h_L)$ by the tensor product $(e_K \otimes f_L)$, where (e_K) and (f_L) denote block bases of the one parameter Haar system, such that $(e_K \otimes f_L)$ satisfies (P1)–(P4) as well as some additional regularity assumptions (see Lemma 3.2 (i) and (ii)). Hence, the conclusions (i)–(iii) of Theorem 4.4 are true for the block basis $(\tilde{b}_R^{(\varepsilon)} : R \in \mathcal{D}^m \times \mathcal{D}^n)$ given by

$$\tilde{b}_R^{(\varepsilon)} = \sum_{K \times L \in \mathcal{B}_R} \varepsilon_{K \times L} e_K \otimes f_L, \quad R \in \mathcal{D}^m \times \mathcal{D}^n.$$

4.4. Local factorization. Here, we state our local factorization result Theorem 4.6, which follows by a standard argument from the projection Theorem 3.1 and the almost-diagonalization result Theorem 4.2. For sake of completeness and since we need to keep track of our constants, we repeat the proof pattern in [7]. For the one-parameter analogue of this proof, we refer to [11, Chapter 5.2].

Theorem 4.6. *Let $1 \leq p, q < \infty$ and $\delta > 0$. For $m, n \in \mathbb{N}$ and $\Gamma, \eta > 0$ there exists an integer $N = N(\delta, m, n, \Gamma, \eta)$ so that the following holds: For any operator $T : H_m^p(H_N^q) \rightarrow H_m^p(H_N^q)$ (respectively $T : H_m^p(H_N^q)^* \rightarrow H_m^p(H_N^q)^*$) with $\|T\| \leq \Gamma$ satisfying*

$$|\langle h_R, Th_R \rangle| \geq \delta |R|, \quad R \in \mathcal{D}^m \times \mathcal{D}^N,$$

the identity Id on $H_m^p(H_n^q)$ (respectively $H_m^p(H_n^q)^$) well factors through T . To be more precise, there exist bounded linear operators $E : H_m^p(H_n^q) \rightarrow H_m^p(H_N^q)$ and $P : H_m^p(H_N^q) \rightarrow H_m^p(H_n^q)$ (respectively $E : H_m^p(H_n^q)^* \rightarrow H_m^p(H_N^q)^*$ and $P : H_m^p(H_N^q)^* \rightarrow H_m^p(H_n^q)^*$) such that the diagram*

$$\begin{array}{ccc} H_m^p(H_n^q) & \xrightarrow{\text{Id}} & H_m^p(H_n^q) \\ E \downarrow & & \uparrow P \\ H_m^p(H_N^q) & \xrightarrow{T} & H_m^p(H_N^q) \end{array} \quad \text{respectively} \quad \begin{array}{ccc} H_m^p(H_n^q)^* & \xrightarrow{\text{Id}} & H_m^p(H_n^q)^* \\ E \downarrow & & \uparrow P \\ H_m^p(H_N^q)^* & \xrightarrow{T} & H_m^p(H_N^q)^* \end{array}$$

is commutative, and the operators E and P can be chosen so that $\|E\|\|P\| \leq (1 + \eta)/\delta$.

Note that for $T = \delta \text{Id}_n$, we have $\|E\|\|P\| = 1/\delta$.

The proof of Theorem 4.6 relies on Theorem 4.2, which builds a block basis $(b_R^{(\varepsilon)})$ of the bi-parameter Haar system that almost-diagonalizes the operator T while simultaneously maintaining the large diagonal: $\langle b_R^{(\varepsilon)}, Tb_R^{(\varepsilon)} \rangle \geq \delta |R|$, $R \in \mathcal{D}^m \times \mathcal{D}^N$. Moreover, $(b_R^{(\varepsilon)})$ satisfies the local product conditions, see Section 3, which implies that $(b_R^{(\varepsilon)})$ is equivalent to the bi-parameter Haar system in $H^p(H^q)$ and $H^p(H^q)^*$, and that the orthogonal projection onto $(b_R^{(\varepsilon)})$ is bounded on $H^p(H^q)$ and on $H^p(H^q)^*$.

Proof of Theorem 4.6. We will only prove the case where $T : H_m^p(H_N^q) \rightarrow H_m^p(H_N^q)$, since the other case is completely analogous. But before we begin with the actual proof, observe that we can assume that

$$\langle h_R, Th_R \rangle \geq \delta |R|, \quad R \in \mathcal{D}^m \times \mathcal{D}^N.$$

Indeed, define $\mathcal{M} : H_m^p(H_N^q) \rightarrow H_m^p(H_N^q)$ as the linear extension of $\mathcal{M}h_R = \text{sign}(\langle h_R, Th_R \rangle)$, $R \in \mathcal{D}^m \times \mathcal{D}^N$. Note that \mathcal{M} is a norm 1 operator and $\langle h_R, T\mathcal{M}h_R \rangle \geq \delta |R|$.

Now let $1 \leq p, q < \infty$, $\delta > 0$, $m, n \in \mathbb{N}$ and $\gamma, \eta > 0$ be fixed. Let $\eta' > 0$ be a small constant satisfying the estimates

$$\eta' \frac{mn}{(1 + \eta')\delta} < 1, \quad \text{and} \quad \frac{(1 + \eta')^{3k}}{\delta - \eta' \frac{mn}{1 + \eta'}} \leq \frac{1 + \eta}{\delta}. \quad (4.51)$$

By Theorem 4.2, we can find an integer $N = N(m, n, \Gamma, \eta')$ so that for any operator $T : H_m^p(H_N^q) \rightarrow H_m^p(H_N^q)$ with $\|T\| \leq \Gamma$, there exist collections $(\mathcal{B}_R : R \in \mathcal{D}^m \times \mathcal{D}^n)$ and signs $(\varepsilon_Q : Q \in \mathcal{B})$ defining a block basis of the Haar system $(b_R^{(\varepsilon)} : R \in \mathcal{D}^m \times \mathcal{D}^n)$ by

$$b_R^{(\varepsilon)} = \sum_{Q \in \mathcal{B}_R} \varepsilon_Q h_Q, \quad R \in \mathcal{D}^m \times \mathcal{D}^n,$$

so that the following conditions are satisfied:

- (a) $\mathcal{B}_R \subset \mathcal{D}^m \times \mathcal{D}^N$, for all $R \in \mathcal{D}^m \times \mathcal{D}^n$.
- (b) $(\mathcal{B}_R : R \in \mathcal{D}^m \times \mathcal{D}^n)$ satisfies the local product conditions (see Section 3) with constants $C_X = 1$ and $C_Y = 1 + \eta'$.
- (c) $(b_R^{(\varepsilon)} : R \in \mathcal{D}^m \times \mathcal{D}^n)$ almost-diagonalizes T so that T has large diagonal. To be more precise, we have the estimates

$$\sum_{\substack{R' \in \mathcal{D}^m \times \mathcal{D}^n \\ R' \neq R}} |\langle b_R^{(\varepsilon)}, T b_{R'}^{(\varepsilon)} \rangle| \leq \eta' \|b_R^{(\varepsilon)}\|_2^2, \quad R \in \mathcal{D}^m \times \mathcal{D}^n, \quad (4.52a)$$

$$\langle b_R^{(\varepsilon)}, T b_R^{(\varepsilon)} \rangle \geq \delta \|b_R^{(\varepsilon)}\|_2^2, \quad R \in \mathcal{D}^m \times \mathcal{D}^n. \quad (4.52b)$$

The rest of the proof is exactly as outlined in [11, Chapter 5.2]. Also see [7] for a specific bi-parameter variant following [11, Chapter 5.2]. Additionally, we will keep track of the exact value of our constants. Define the subspace Y of $H_m^p(H_N^q)$ (see condition (a)) by

$$Y = \text{span}\{b_R^{(\varepsilon)} : R \in \mathcal{D}^m \times \mathcal{D}^n\},$$

equipped with the $H^p(H^q)$ norm. Condition (b) has three implications that we will now record. Firstly, for any $1 \leq r, s < \infty$ and $1 < r', s' \leq \infty$ with $\frac{1}{r} + \frac{1}{r'} = 1$ and $\frac{1}{s} + \frac{1}{s'} = 1$, we have by Theorem 3.1 that the operator $E : H_m^p(H_n^q) \rightarrow Y$ defined as the linear extension of $h_R \mapsto b_R^{(\varepsilon)}$, $R \in \mathcal{D}^m \times \mathcal{D}^n$, satisfies

$$\begin{array}{ccc} H_m^p(H_n^q) & \xrightarrow{\text{Id}} & H_m^p(H_n^q) \\ E \downarrow & & \uparrow E^{-1} \\ Y & \xrightarrow{\text{Id}} & Y \end{array} \quad \|E\| \|E^{-1}\| \leq (1 + \eta')^{2k}, \quad (4.53)$$

where k is the integer in Theorem 3.1. Thirdly, by (4.52b) together with the projection Theorem 3.1, we obtain that the operator $U : H^p(H^q) \rightarrow Y$, defined by

$$Uf = \sum_{R \in \mathcal{D}^m \times \mathcal{D}^n} \frac{\langle b_R^{(\varepsilon)}, f \rangle}{\langle b_R^{(\varepsilon)}, T b_R^{(\varepsilon)} \rangle} b_R^{(\varepsilon)}, \quad f \in H^p(H^q),$$

satisfies the estimate

$$\|Uf\|_{H^p(H^q)} \leq \frac{(1 + \eta')^k}{\delta} \|f\|_{H^p(H^q)}, \quad f \in H^p(H^q). \quad (4.54)$$

For $g = \sum_{R \in \mathcal{D}^m} a_R b_R^{(\varepsilon)} \in Y$, Theorem 3.1 together with (4.52) yields that

$$\|UTg - g\|_{H^p(H^q)} \leq \eta' \frac{mn}{(1 + \eta')\delta} \|g\|_{H^p(H^q)}. \quad (4.55)$$

Let $J : Y \rightarrow H_m^p(H_N^q)$ denote the operator given by $Jy = y$. Define the operator $V : H_m^p(H_N^q) \rightarrow Y$ by $V = (UTJ)^{-1}U$ (which is well defined by (4.51)), and note that

$$\begin{array}{ccc}
 Y & \xrightarrow{\text{Id}} & Y \\
 \downarrow J & \nearrow (UTJ)^{-1} & \uparrow V \\
 & Y & \\
 & \nwarrow U & \\
 H_m^p(H_N^q) & \xrightarrow{T} & H_m^p(H_N^q)
 \end{array}
 \quad \|J\|\|V\| \leq \frac{(1 + \eta')^k}{\delta - \eta' \frac{mn}{1 + \eta'}}. \quad (4.56)$$

Merging diagram (4.53) with diagram (4.56) and recalling (4.51) concludes the proof. \square

Remark 4.7. Similar to Remark 4.3 (see also Remark 3.3), we could replace the bi-parameter Haar system $(h_K \otimes h_L)$ in Theorem 4.6 with a tensor product $(e_K \otimes f_L)$ that satisfies (P1)–(P4) and some additional regularity assumptions (see Lemma 3.2 (i) and (ii)), and simply repeat the proof. To be more precise, the large diagonal hypothesis of Theorem 4.6 would read as follows:

$$|\langle e_K \otimes f_L, T e_K \otimes f_L \rangle| \geq \delta \|e_K \otimes f_L\|_2^2, \quad K \in \mathcal{D}^m, L \in \mathcal{D}^N,$$

respectively $K \in \mathcal{D}^M, L \in \mathcal{D}^n$.

5. SUMS OF FINITE DIMENSIONAL BANACH SPACES

Section 5.1, we discuss the necessary tools to diagonalize operators acting on a direct sum of finite dimensional Banach spaces. In Section 5.2, we describe how to “glue together” factorization results in finite dimensional Banach spaces, to obtain a factorization result in the direct sum of these spaces. The proofs of the theorems in Section 5.1 and Section 5.2 have been repeated in numerous situations see e.g. [3, 2, 11, 12, 7]. This is the author’s attempt to avoid repetition in upcoming papers. In Section 5.3 we discuss isomorphisms and non-isomorphisms of direct sums of finite dimensional Banach spaces. Finally, we give proofs of the main results Theorem 2.1 and Theorem 2.2 in Section 5.4 and Section 5.5, respectively.

5.1. Diagonalization.

We briefly discuss two lemmas to diagonalize an operator on a direct sum of finite dimensional Banach spaces. The first lemma follows by a gliding hump argument, and is therefore limited to finite parameters in the direct sum. The second lemma for direct sums with infinite parameter, uses an additional hypothesis, see Definition 5.2. For the space $(\sum_{n \in \mathbb{N}} H_n^p(H_n^q))_\infty$, this hypothesis will be realized by Theorem 4.4.

Lemma 5.1. *Let $1 \leq r < \infty$, and let $(X_n)_{n \in \mathbb{N}}$ be a non-decreasing sequence of finite dimensional Banach spaces. Let $X^{(r)} = (\sum_{n \in \mathbb{N}} X_n)_r$ and $T : X^{(r)} \rightarrow X^{(r)}$ be a bounded linear operator. For each $\theta > 0$ there exist norm 1 operators $U, V : X^{(r)} \rightarrow X^{(r)}$ such that $UV = \text{Id}_{X^{(r)}}$, and \hat{T} given by $\hat{T} = UTV$ is almost diagonal, i.e.*

$$\|\hat{T} - \sum_{n=1}^{\infty} P_n \hat{T} P_n\| \leq \theta. \quad (5.1)$$

The norm 1 operator $P_n : X^{(r)} \rightarrow X^{(r)}$ denotes the coordinate projection onto X_n . The above series of operators is understood as a formal series and does not indicate any form of convergence.

We remark that an operator $D : X^{(r)} \rightarrow X^{(r)}$ is called *diagonal operator* if $D = \sum_{n=1}^{\infty} P_n D P_n$. The proof of Lemma 5.1 is a standard gliding hump argument and therefore omitted.

Definition 5.2 is merely a surrogate of the corresponding theorems in [3, 2, 11, 12, 7].

Definition 5.2. We say that a non-decreasing sequence of finite dimensional Banach spaces $(X_n)_{n \in \mathbb{N}}$ with $\sup_n \dim X_n = \infty$ has the property that *projections almost annihilate finite dimensional subspaces with constant $C_P > 0$* if the following conditions are satisfied:

For all $n, d \in \mathbb{N}$ and $\eta > 0$ there exists an integer $N = N(n, d, \eta)$ such that for any d -dimensional subspace $F \subset X_N$ there exists a bounded projection $Q : X_N \rightarrow X_N$ and an isomorphism $S : X_n \rightarrow Q(X_N)$ such that

- (i) $\|Q\| \leq C_P$,
- (ii) $\|S\|, \|S^{-1}\| \leq C_P$,
- (iii) $\|Qx\| \leq \eta\|x\|$, for all $x \in F$.

The following diagonalization Lemma 5.3 allows us to diagonalize an operator on direct sums with infinite parameter $r = \infty$, by additionally using the property defined in Definition 5.2.

Lemma 5.3. Let $(X_n)_{n \in \mathbb{N}}$ denote a non-decreasing sequence of finite dimensional Banach spaces with $\sup_n \dim X_n = \infty$ having the property that *projections almost annihilate finite dimensional subspaces with constant $C_P > 0$* (see Definition 5.2). Now put $X^{(\infty)} = (\sum_{n \in \mathbb{N}} X_n)_{\infty}$ and let $T : X^{(\infty)} \rightarrow X^{(\infty)}$ be a bounded linear operator. For each $\theta > 0$ there exist operators $U, V : X^{(\infty)} \rightarrow X^{(\infty)}$ such that $UV = \text{Id}_{X^{(\infty)}}$, and \hat{T} given by $\hat{T} = UTV$ is almost diagonal, i.e.

$$\|\hat{T} - \sum_{n=1}^{\infty} P_n \hat{T} P_n\| \leq \theta. \quad (5.2)$$

The norm 1 operator $P_n : X^{(\infty)} \rightarrow X^{(\infty)}$ denotes the coordinate projection onto X_n . The above series of operators is understood as a formal series and does not indicate any form of convergence. The operators U and V can be chosen such that $\|U\|\|V\| \leq C_P^3$.

The proof is completely analogous to the corresponding diagonalization theorems in [3, 2, 11, 12, 7], and we therefore omit it.

5.2. Glueing.

Here, we “glue together” factorization diagrams for finite dimensional Banach spaces, to obtain a factorization diagram for the direct sum of these spaces. We distinguish between finite and infinite parameters. Again, we refer to [3, 2, 11, 12, 7].

Proposition 5.4. Let $(X_n)_{n \in \mathbb{N}}$ be an increasing sequence of finite dimensional Banach spaces. Let $\Gamma > 0$ and $\eta > 0$ be fixed. Assume that for each $n \in \mathbb{N}$ there exists an integer $N = N(n, \Gamma, \eta)$ such that for any operator $T_n : X_n \rightarrow X_N$ with $\|T_n\| \leq \Gamma$ one can find operators $R_n : X_n \rightarrow X_N$ and $S_n : X_N \rightarrow X_n$ so that

$$\begin{array}{ccc} X_n & \xrightarrow{\text{Id}_{X_n}} & X_n \\ S_n \downarrow & & \uparrow R_n \\ X_N & \xrightarrow{H_n} & X_N \end{array} \quad (5.3)$$

where $H_n = T_n$ or $H_n = \text{Id}_{X_N} - T_n$, and $\|R_n\|\|S_n\| \leq 1 + \eta$.

Let $1 \leq r \leq \infty$, put $X^{(r)} = (\sum_{n \in \mathbb{N}} X_n)_r$ and let $T : X^{(r)} \rightarrow X^{(r)}$ be a bounded, linear operator with $\|T\| \leq \Gamma$. If $r = \infty$ (and only then) we assume additionally that $(X_n)_{n \in \mathbb{N}}$ has the property that projections almost annihilate finite dimensional subspaces with constant $C_P > 0$ (see Definition 5.2).

Then there exist operators $P, Q : X^{(r)} \rightarrow X^{(r)}$ such that

$$\begin{array}{ccc} X^{(r)} & \xrightarrow{\text{Id}_{X^{(r)}}} & X^{(r)} \\ P \downarrow & & \uparrow Q \\ X^{(r)} & \xrightarrow{H} & X^{(r)} \end{array} \quad (5.4)$$

for $H = T$ or $H = \text{Id}_{X^{(r)}} - T$. For each $\varepsilon > 0$ the operators P and Q can be chosen so that $\|P\|\|Q\| \leq 1 + \eta + \varepsilon$, if $r < \infty$, and for $r = \infty$ we obtain $\|P\|\|Q\| \leq C_P^3(1 + \eta + \varepsilon)$.

Proof. For a proof see e.g. [2, 7]. \square

5.3. Isomorphisms and non-isomorphisms.

Here, we briefly discuss two results on sums of finite dimensional Banach spaces. Together, they show us that $(\sum_{m,n} H_m^p(H_n^q))_r$ is isomorphic to $(\sum_n H_n^p(H_n^q))_s$ if and only if $r = s$. The same is true for $(\sum_{m,n} H_m^p(H_n^q)^*)_r$ and $(\sum_n H_n^p(H_n^q)^*)_s$.

The following Proposition 5.5 is a simple consequence of Pelczyński's decomposition method. Therefore, we omit the proof.

Proposition 5.5. *Let $1 \leq r \leq \infty$ and let $(X_{m,n} : m, n \in \mathbb{N})$ denote a sequence of finite dimensional Banach spaces such that $X_{m,n} \subset X_{m+1,n}$ and $X_{m,n} \subset X_{m,n+1}$. Then the space $(\sum_{m,n \in \mathbb{N}} X_{m,n})_r$ is isometrically isomorphic to $(\sum_{n \in \mathbb{N}} X_{n,n})_r$.*

Theorem 5.6 is a finite dimensional Banach space variant of Pitt's theorem. See also [5].

Theorem 5.6. *Let $1 \leq r, s \leq \infty$, and let $(X_n)_{n \in \mathbb{N}}$ denote an increasing sequence of finite dimensional Banach spaces. Let $X^{(r)}$ denote $(\sum_{n \in \mathbb{N}} X_n)_r$ and let $X^{(s)}$ denote the space $(\sum_{n \in \mathbb{N}} X_n)_s$. If $T : X^{(r)} \rightarrow X^{(s)}$ is an isomorphism, then $r = s$. Consequently, all the spaces $X^{(r)}$, $1 \leq r \leq \infty$ are mutually non-isomorphic.*

The proof is a standard gliding hump argument for $1 \leq r, s < \infty$. The remaining cases follow immediately by considering the separability/non-separability of the respective spaces. For those reasons, we omit the proof.

5.4. Proof of the main result Theorem 2.1.

For convenience, we reassert Theorem 2.1 here.

Theorem 5.7 (Main result Theorem 2.1). *Let $1 \leq p, q < \infty$ and $1 \leq r \leq \infty$, and for all $n \in \mathbb{N}$ let X_n denote the space $H_n^p(H_n^q)$ or its dual $H_n^p(H_n^q)^*$. For any $\eta > 0$ and any operator $T : (\sum_{n \in \mathbb{N}} X_n)_r \rightarrow (\sum_{n \in \mathbb{N}} X_n)_r$, there exist operators $R, S : (\sum_{n \in \mathbb{N}} X_n)_r \rightarrow (\sum_{n \in \mathbb{N}} X_n)_r$ such that*

$$\begin{array}{ccc} (\sum_{n \in \mathbb{N}} X_n)_r & \xrightarrow{\text{Id}} & (\sum_{n \in \mathbb{N}} X_n)_r \\ S \downarrow & & \uparrow R \\ (\sum_{n \in \mathbb{N}} X_n)_r & \xrightarrow{H} & (\sum_{n \in \mathbb{N}} X_n)_r \end{array} \quad (5.5)$$

for $H = T$ or $H = \text{Id} - T$ and $\|R\|\|S\| \leq 2 + \eta$.

The following Ramsey type Theorem 5.8 is the last missing ingredient for the proof of Theorem 5.7 (Main result Theorem 2.1).

Theorem 5.8. *Given $n_0 \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that for any collection $\mathcal{C} \subset \mathcal{D}^n \times \mathcal{D}^n$ one finds $\mathcal{A}, \mathcal{B} \subset \mathcal{D}$ satisfying*

- (i) $\mathcal{A} \times \mathcal{B} \subset \mathcal{C}$ or $\mathcal{A} \times \mathcal{B} \subset (\mathcal{D}^n \times \mathcal{D}^n) \setminus \mathcal{C}$,
- (ii) $[\mathcal{A}] \geq n_0$ and $[\mathcal{B}] \geq n_0$.

One can choose $n = n_0 2^{4^{n_0}}$.

Proof. We refer to [10]. See also [7]. □

Proof of Theorem 5.7 (Main result Theorem 2.1). The proof follows the pattern of the corresponding proof in [7]. Let $1 \leq p, q < \infty$, $1 \leq r \leq \infty$ and define the space $X^{(r)}$ by

$$X^{(r)} = \left(\sum_{n \in \mathbb{N}} X_n \right)_r.$$

Let $\eta > 0$ and $T : X^{(r)} \rightarrow X^{(r)}$. Again, we will only prove the case where $X_n = H_n^p(H_n^q)$. The case $X_n = H_n^p(H_n^q)^*$ is repeating the following argument with the roles of T and T^* reversed.

In the first part of the proof we will show that for all $n \in \mathbb{N}$ and $\Gamma > 0$, there is an integer $N = N(n, \Gamma, \eta)$ such that for any operator $D_n : X_N \rightarrow X_N$ with $\|D_n\| \leq \Gamma$ there exist operators R_n, S_n so that

$$\begin{array}{ccc} X_n & \xrightarrow{\text{Id}} & X_n \\ S_n \downarrow & & \uparrow R_n \\ X_N & \xrightarrow{H_n} & X_N \end{array} \quad \|R_n\| \|S_n\| \leq 2 + \eta, \quad (5.6)$$

where $H_n = D_n$ or $H_n = \text{Id}_{X_N} - D_n$.

To this end, let $\eta' > 0$ be parameter, which will be specified at a later point. Firstly, we choose $\gamma = \gamma(n, \eta')$ so large, so that for any collection $\mathcal{E} \subset \mathcal{D}$ with Carleson constant $[\mathcal{E}] \geq \gamma$ exist collections $\mathcal{E}_I \subset \mathcal{E}$, $I \in \mathcal{D}^n$, and an affine map $\psi : [0, 1) \rightarrow [0, 1)$ so that the sequence of collections $(\psi(\mathcal{E}_I) \times \psi(\mathcal{E}_J) : I, J \in \mathcal{D}^n)$ satisfies (P1)–(P4) with constant $C_X = 1 + \eta'$, as well as the additional regularity assumptions (i) and (ii) of Lemma 3.2. For a detailed exposition we refer the reader to [11].

Secondly, if we put $n_1 = \lceil \gamma 2^{4^\gamma} \rceil$, the Ramsey Theorem 5.8 asserts that whenever $\mathcal{C} \subset \mathcal{D}^{n_1} \times \mathcal{D}^{n_1}$, there exist collections $\mathcal{E}, \mathcal{F} \subset \mathcal{D}^{n_1}$ with $[\mathcal{E}], [\mathcal{F}] \geq \gamma$ so that either

$$\mathcal{E} \times \mathcal{F} \subset \mathcal{C} \quad \text{or} \quad \mathcal{E} \times \mathcal{F} \subset \mathcal{D}^{n_1} \times \mathcal{D}^{n_1} \setminus \mathcal{C}.$$

Thirdly, applying Theorem 4.2 with $\delta = 0$, yields an integer $N = N(n_1, \Gamma, \eta')$ (this is exactly the integer N of Theorem 4.2 with the specified parameters) and a sequence of collections of sets $(\mathcal{B}_R : R \in \mathcal{D})$ with the following properties:

- (a) $\mathcal{B}_R \subset \mathcal{D}^N \times \mathcal{D}^N$ for all $R \in \mathcal{D}^{n_1} \times \mathcal{D}^{n_1}$.
- (b) $(\mathcal{B}_R : R \in \mathcal{D}^n \times \mathcal{D}^n)$ satisfies the local product conditions with constants $C_X = 1 + \eta'$ and $C_Y = 1 + \eta'$.
- (c) The $(b_R : R \in \mathcal{D}^n \times \mathcal{D}^n)$ almost-diagonalize D_n . To be more precise, we have the estimate

$$\sum_{\substack{R' \in \mathcal{D}^n \times \mathcal{D}^n \\ R' \neq R}} |\langle b_R, D_n b_{R'} \rangle| \leq \eta' \|b_R\|_2^2, \quad R \in \mathcal{D}^n \times \mathcal{D}^n. \quad (5.7)$$

We note that since there is no lower estimate for the diagonal, we can choose all the signs ε_Q equal to 1, so henceforth we will omit the superscript (ε) of $b_R^{(\varepsilon)}$, and simply denote the function by b_R .

Fourthly, we will now combine the first three steps. We specify the collection of dyadic rectangles \mathcal{C} by

$$\mathcal{C} = \{R \in \mathcal{D}^{n_1} \times \mathcal{D}^{n_1} : |\langle b_R, D_n b_R \rangle| \geq \|b_R\|_2^2/2\}. \quad (5.8)$$

By the choice of our parameters in the first two steps, we can find finite sequences of collections $(\mathcal{E}_I : I \in \mathcal{D}^n)$ and $(\mathcal{F}_J : J \in \mathcal{D}^n)$ so that

$$\mathcal{E}_I \times \mathcal{F}_J \subset \mathcal{C} \quad \text{or} \quad \mathcal{E}_I \times \mathcal{F}_J \subset \mathcal{D}^{n_1} \times \mathcal{D}^{n_1} \setminus \mathcal{C},$$

for all $I, J \in \mathcal{D}^{n_1}$. If the first inclusion is true we put $H_n = D_n$, if the second is true, then we define $H_n = \text{Id}_{X_N} - D_n$. We will now construct a block basis (\tilde{b}_R) of the block basis (b_R) of the Haar system (h_R) . We define the collection of dyadic rectangles $\tilde{\mathcal{B}}_{I \times J}$ by

$$\tilde{\mathcal{B}}_{I \times J} = \bigcup_{\substack{E \in \mathcal{E}_I \\ F \in \mathcal{F}_J}} \mathcal{B}_{E \times F}, \quad I, J \in \mathcal{D}^n, \quad (5.9)$$

and the corresponding block basis elements $\tilde{b}_{I \times J}$ by

$$\tilde{b}_R = \sum_{Q \in \tilde{\mathcal{B}}_R} h_Q, \quad R \in \mathcal{D}^n \times \mathcal{D}^n. \quad (5.10)$$

Note that $\tilde{\mathcal{B}}_R \subset \mathcal{D}^N \times \mathcal{D}^N$, $R \in \mathcal{D}^n \times \mathcal{D}^n$ hence $\tilde{b}_R \in H_N^p(H_N^q)$, $R \in \mathcal{D}^n \times \mathcal{D}^n$. The reiteration Lemma 3.2 gives us that $\tilde{\mathcal{B}}_R$, $R \in \mathcal{D}^n \times \mathcal{D}^n$ satisfies the local product conditions (P1)–(P4) with constants $C_X = C_Y = (1 + \eta')^4$. Now put

$$Y_n = \text{span}\{\tilde{b}_R : R \in \mathcal{D}^n \times \mathcal{D}^n\} \subset X_N,$$

equipped with the $H^p(H^q)$ norm. We summarize what we have proved this far: by Theorem 3.1, we have that

$$\begin{array}{ccc} X_n & \xrightarrow{\text{Id}_{X_n}} & X_n \\ E_n \downarrow & & \uparrow E_n^{-1} \\ Y_n & \xrightarrow{\text{Id}_{Y_n}} & Y_n \end{array} \quad \|E_n\| \|E_n^{-1}\| \leq (1 + \eta')^{8k}, \quad (5.11)$$

where k is the integer appearing in Theorem 3.1. Furthermore, by (5.7), (5.8), (5.9), (5.10) and Theorem 3.1 we have the estimates

$$\sum_{\substack{R' \in \mathcal{D}^n \times \mathcal{D}^n \\ R' \neq R}} |\langle \tilde{b}_R, H_n \tilde{b}_{R'} \rangle| \leq c(n, \eta') \|\tilde{b}_R\|_2^2, \quad R \in \mathcal{D}^n \times \mathcal{D}^n, \quad (5.12a)$$

$$|\langle \tilde{b}_R, H_n \tilde{b}_R \rangle| \geq \left(\frac{1}{2} - c(n, \eta')\right) \|\tilde{b}_R\|_2^2, \quad R \in \mathcal{D}^n \times \mathcal{D}^n, \quad (5.12b)$$

where $c(n, \eta') \rightarrow 0$, if $\eta' \rightarrow 0$. Remark Remark 4.7 allows us to replace $b_R^{(\varepsilon)}$ in Theorem 4.6 by \tilde{b}_R , thus Theorem 4.6 yields

$$\begin{array}{ccc} Y_n & \xrightarrow{\text{Id}_{Y_n}} & Y_n \\ E \downarrow & & \uparrow P \\ H_m^p(H_N^q) & \xrightarrow{H_n} & H_m^p(H_N^q) \end{array}$$

shows that (5.6) is true, if we choose η' appropriately. Alternatively to invoking Remark 4.7 and Theorem 4.6, we could repeat the proof of Theorem 4.6 after (4.52) with \tilde{b}_R instead of $b_R^{(\varepsilon)}$ (which is of course part of the argument behind Remark 4.7).

Finally, observe that Theorem 4.4 implies that $(X_n)_{n \in \mathbb{N}}$ has the property that projections almost annihilate finite dimensional subspaces with constant $1 + \eta'$ (see Definition 5.2). Thus, applying Proposition 5.4 concludes the proof. \square

5.5. Proof of the main result Theorem 2.2.

We will now give the proof Theorem 2.2. It follows from Theorem 2.1 and Pełczyński's decomposition method.

Proof. Let $1 \leq r \leq \infty$ and let X denote either the space $(\sum_{n \in \mathbb{N}} H_n^p(H_n^q))_r$ or $(\sum_{n \in \mathbb{N}} H_n^p(H_n^q)^*)_r$. Let Q be a bounded projection on X .

Clearly, $(\sum X)_r$ is isomorphic to $(\sum(\sum X)_r)_r$. Furthermore, X is isomorphic to a complemented subspace of $(\sum X)_r$, and $(\sum X)_r$ is isomorphic to a complemented subspace of X . Hence, by Pełczyński's decomposition method, X is isomorphic to $(\sum X)_r$. From Theorem 2.1, we obtain that

$$\begin{array}{ccc} X & \xrightarrow{\text{Id}} & X \\ S \downarrow & & \uparrow R \\ X & \xrightarrow{H} & X \end{array}$$

where $H = Q$ or $H = \text{Id} - Q$. The diagram shows that $H(X)$ is a complemented subspace of X , and that X is isomorphic to a complemented subspace of $H(X)$, hence, by Pełczyński's decomposition method we obtain that $H(X)$ is isomorphic to X . \square

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